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REGULARIZED REDUCED ORDER MODELS FOR A STOCHASTIC BURGERS EQUATION

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Dedicated to Professor William J. Layton on the occasion of his 60^{th} birthday

Abstract. In this paper, we study the numerical stability of reduced order models for convection-dominated stochastic systems in a relatively simple setting: a stochastic Burgers equation with linear multiplicative noise. Our preliminary results suggest that, in a convection-dominated regime, standard reduced order models yield inaccurate results in the form of spurious numerical oscillations. To alleviate these oscillations, we use the Leray reduced order model, which increases the numerical stability of the standard model by smoothing (regularizing) the convective term with an explicit spatial filter. The Leray reduced order model yields significantly better results than the standard reduced order model and is more robust with respect to changes in the strength of the noise.

Key words. Reduced order modeling, Leray regularized model, stabilization method, numerical instability, stochastic Burgers equation, differential filter

1. Introduction

Reduced order models (ROMs) are commonly used in applications that require repeated numerical simulations of large, complex systems [35, 55]. ROMs have been successful in the numerical simulation of various fluid flows [37, 52]. Numerical instability, usually in the form of unphysical numerical oscillations, is one of the main challenges for ROMs of fluid flows described by the Navier-Stokes equations (NSE). There are several sources of numerical instability of ROMs for fluid flows [16], such as (i) the convection-dominated (high Reynolds number) regime, in which the convection nonlinear term plays a central role [3, 37, 52]; and (ii) the inf-sup condition, which imposes a constraint on the ROM velocity and pressure spaces [5, 16]. To mitigate the spurious numerical oscillations created by these sources of numerical instability, various stabilized ROMs have been proposed (see, e.g., [2, 3, 4, 5, 6, 8, 17, 26, 33, 40, 54, 56, 62, 64] for such examples). A promising recent development in this class of methods is regularized ROMs [59, 63], which use explicit spatial filtering to increase the numerical stability of the ROM approximation.

Recently, the development of ROMs for systems involving random components has also received increased attention. For instance, ROMs for partial differential equations (PDEs) subject to random inputs acting on the boundary as well as PDEs with random coefficients have been considered in various contexts [22, 24, 29, 44, 13, 14, 23, 34, 61]. However, ROMs for evolutionary PDEs driven by stochastic processes such as Brownian motion seem to be much less investigated. To our knowledge, only a few works are available [15]; see also [20], where a new stochastic

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parameterization framework is presented to address a related question of parameterizing the unresolved high-frequency modes in terms of the resolved low-frequency modes.

In this article, we consider ROMs within the context of nonlinear stochastic PDEs (SPDEs) that are of relevance to fluid dynamics. The main purpose is to investigate within a simple relevant setting—a stochastic Burgers equation (SBE) driven by linear multiplicative noise—the stabilization of the standard Galerkin ROM (G-ROM) in a convection-dominated regime. It is numerically illustrated that spurious oscillations developed in a G-ROM persist as the noise is turned on, and the oscillations worsen as the noise amplitude increases. A Leray regularized ROM (referred to as L-ROM hereafter) is then tested. The L-ROM provides more accurate modeling of the SBE dynamics by greatly reducing the artificial oscillations of the G-ROM, especially when the ROM dimension is low; cf. Figs. 3–6. It is further illustrated that the L-ROM is much more robust than the G-ROM with respect to the noise amplitude as revealed by the statistics of the corresponding modeling errors, which have significantly lower mean and variance; cf. Fig. 7.

The rest of the paper is organized as follows. In Section 2, we outline the SBE to be used in our numerical exploration and derive the corresponding G-ROM and the L-ROM based on the proper orthogonal decomposition. The performance of the two ROMs is then tested and compared in Section 3 by placing the SBE in a convection-dominated regime. Finally, some concluding remarks and potential future research directions are given in Section 4.

2. Reduced Order Models for a Stochastic Burgers Equation

The viscous Burgers equation and its stochastic versions have been used previously to test new techniques in reduced order modeling and related contexts; see among many others [20, 19, 42, 43, 18, 51]. In this paper, we focus on an SBE driven by linear multiplicative noise, which is presented briefly in Section 2.1. To fix ideas, the ROMs explored in this paper are derived based on the proper orthogonal decomposition. In Section 2.2, we outline the main steps in the derivation of the proper orthogonal decomposition basis. The standard Galerkin ROM for the SBE is then derived in Section 2.3. In Section 2.4, we develop the Leray ROM, which is a regularized ROM that aims at increasing the numerical stability of the standard ROM for the SBE.

2.1. Stochastic Burgers Equation (SBE). In this paper, we focus on the following stochastic Burgers equation (SBE) driven by linear multiplicative noise:

$$\begin{aligned} &\mathrm{d}u = \big(\nu u_{xx} - u u_x\big)\mathrm{d}t + \sigma u \circ \mathrm{d}W_t, \\ &u(0,t) = u(1,t) = 0, \quad t \geq 0, \\ &u(x,0) = u_0(x), \qquad x \in (0,1), \end{aligned} \tag{1}$$

where ν is a positive diffusion coefficient, W_t is a two-sided one-dimensional Wiener process, σ is a positive constant which measures the "amplitude" of the noise, and u_0 is some appropriate initial datum to be specified below. To fix ideas, the multiplicative noise term $\sigma u \circ \mathrm{d}W_t$ is understood in the sense of Stratonovich [53].

SPDEs driven by linear multiplicative noise such as the SBE (1) arise in various contexts, including turbulence theory or non-equilibrium phase transitions [9, 27, 50], modeling of randomly fluctuating environment [7] in spatially-extended harvesting models [36, 21, 57, 58, 49, 48], or simply modeling of parameter disturbances [10].