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A HIGHER-ORDER ENSEMBLE/PROPER ORTHOGONAL DECOMPOSITION METHOD FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS

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Dedicated to Professor William J. Layton on the occasion of his 60^{th} birthday

Abstract. Partial differential equations (PDE) often involve parameters, such as viscosity or density. An analysis of the PDE may involve considering a large range of parameter values, as occurs in uncertainty quantification, control and optimization, inference, and several statistical techniques. The solution for even a single case may be quite expensive; whereas parallel computing may be applied, this reduces the total elapsed time but not the total computational effort. In the case of flows governed by the Navier-Stokes equations, a method has been devised for computing an ensemble of solutions. Recently, a reduced-order model derived from a proper orthogonal decomposition (POD) approach was incorporated into a first-order accurate in time version of the ensemble algorithm. In this work, we expand on that work by incorporating the POD reduced order model into a second-order accurate ensemble algorithm. Stability and convergence results for this method are updated to account for the POD/ROM approach. Numerical experiments illustrate the accuracy and efficiency of the new approach.

Key words. Navier-Stokes equations, ensemble computation, proper orthogonal decomposition, finite element methods

1. Introduction

In science and engineering, mathematical models are utilized to understand and predict the behavior of complex systems. Two common input data/parameters for these types of models include forcing terms and initial conditions. Often, for any number of reasons, there is a degree of uncertainty involved with the specification of such inputs. In order to obtain an accurate model we must incorporate such uncertainties into the governing equations and quantify their effects on the outputs of the simulation.

In this uncertainty quantification setting, one needs to determine many realizations of the outputs of the simulation in order to determine accurate statistical information about those outputs. A similar need occurs in other settings such as inference, optimization, and control and the need to determine solution ensembles arising in many applications, e.g., weather forecasting and turbulence modeling. Thus, in this work, we are interested in computing ensembles of solutions for the Navier-Stokes equations (NSE) with uncertainty present in the initial conditions

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and body forces. Specifically, for $j = 1, \dots, J$, we have

$$\begin{cases} u_t^j + u^j \cdot \nabla u^j - \nu \triangle u^j + \nabla p^j = f^j(x,t) & \forall x \in \Omega \times (0,T] \\ \nabla \cdot u^j = 0 & \forall x \in \Omega \times (0,T] \\ u^j = 0 & \forall x \in \partial\Omega \times (0,T] \\ u^j(x,0) = u^{j,0}(x) & \forall x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is an open regular domain. In most settings, in order to guarantee a desired accuracy level in the outputs, a fine spatial resolution is usually required, which renders each realization to be computationally intensive. Traditionally, the simulation for each ensemble member is treated as a separate problem; therefore the focus has been to cut down on the total number of realizations needed. However, in recent works [11, 12, 13, 20], new algorithms were designed that allow for the realizations to be computed simultaneously at each time step. In those papers, the focus has been on creating algorithms which allow for the same linear system to be used for all right-hand sides. Hence, the need to solve a different linear system for each right-hand side is reduced to solving the a single system with many different right-hand sides. This is a well studied problem for which efficient block iterative methods already exist. A few examples of these include block CG [2], block QMR [3], and block GMRES [4].

The next natural step to take to further improve upon the efficiency of these ensemble algorithms is the introduction of reduced-order modeling (ROM) techniques. Specifically, we are interested in the implementation of the proper orthogonal decomposition (POD) approach into the ensemble framework. POD works by generally extracting from highly accurate numerical simulations or experimental data the most energetic modes in a given system.

One can then project the original problem onto these POD modes to obtain the Galerkin proper orthogonal decomposition (POD-G-ROM) approximation to the original problem. It has been shown for laminar flows [1, 10] that it is possible to obtain a good approximation with few POD modes, hence the corresponding POD-G-ROM only requires the solution of a small linear system. Recently, in [6], a POD-G-ROM ensemble algorithm based on the first-order accurate in time ensemble algorithm first introduced in [13] was developed. However, in applications that require long-term time integration, such as climate and ocean forecasting, higher-order methods are highly desirable. For this reason, in this paper we expand on the work of [6] by introducing the POD method into the second-order ensemble algorithm introduced in [11].

2. Notation and preliminaries

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the $L^2(\Omega)$ norm and inner product, respectively, and by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$ the $L^p(\Omega)$ and Sobolev $W_p^k(\Omega)$ norms, respectively. $H^k(\Omega) = W_2^k(\Omega)$ with norm $\|\cdot\|_k$. For a function $v(x,t)$ that is well defined on $\Omega \times [0,T]$, we define the norms

$$|||v|||_{2,s} := \Big(\int_0^T \|v(\cdot,t)\|_s^2 dt\Big)^{\frac{1}{2}} \qquad \text{and} \qquad |||v|||_{\infty,s} := \operatorname{ess\,sup}_{[0,T]} \|v(\cdot,t)\|_s.$$

The space $H^{-1}(\Omega)$ denotes the dual space of bounded linear functionals defined on $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$; this space is equipped with the norm

$$\|f\|_{-1} = \sup_{0 \neq v \in X} \frac{(f,v)}{\|\nabla v\|} \quad \forall f \in H^{-1}(\Omega).$$