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USING RBF-GENERATED QUADRATURE RULES TO SOLVE NONLOCAL ANOMALOUS DIFFUSION

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Dedicated to Professor William J. Layton on the occasion of his 60^{th} *birthday*

Abstract. The goal of this work is to solve nonlocal diffusion and anomalous diffusion problems by approximating the nonlocal integral appearing in the integro-differential equation by novel quadrature rules. These quadrature rules are derived so that they are exact for a nonlocal integral evaluated at translations of a given radial basis function (RBF). We first illustrate how to derive RBF-generated quadrature rules in one dimension and demonstrate their accuracy for approximating a nonlocal integral. Once the quadrature rules are derived as a preprocessing step, we apply them to approximate the nonlocal integral in a nonlocal diffusion problem and when the temporal derivative is approximated by a standard difference approximation a system of difference equations are obtained. This approach is extended to two dimensions where both a circular and rectangular nonlocal neighborhood are considered. Numerical results are provided and we compare our results to published results solving nonlocal problems using standard finite element methods.

Key words. Nonlocal, anomalous diffusion, radial basis functions, RBF, quadrature

1. Introduction

In recent years, there has been an increased interest in nonlocal continuum models due to their ability to describe physical phenomena which are not well modeled by standard partial differential equation (PDE) models. Unlike standard PDE models, nonlocal models are free of spatial derivatives. Feature interactions are typically represented by an integral resulting in an integro-differential equation; these interactions are assumed to occur over a finite region governed by a horizon. Nonlocal models for anomalous diffusion are especially advantageous because the only difference in the model is the exponent in the kernel of the nonlocal integral. Thus methods for solving a nonlocal diffusion problem typically can be easily extended to model anomalous diffusion unlike PDE models. One complication in nonlocal models is that the region of integration for the nonlocal integral can extend past the physical domain. Thus so-called volume constraints are imposed in place of the standard boundary conditions in PDE models. For an overview of the analysis of nonlocal problems with volume constraints the reader is referred to [1].

In this paper we are interested in nonlocal diffusion and especially in the case of anomalous diffusion where the spatial spread of a diffusing quantity is not proportional to the square root of time as predicted by the heat equation. Several authors have investigated the numerical solution of such linear time dependent problems. For example, in [2, 3] the authors use standard finite element methods to approximate the nonlocal problem when the solution is continuous and discontinuous Galerkin for a discontinuous solution.

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The computational cost of solving a nonlocal problem is often higher than solving a local problem. One difficulty is that the bandwidth of the resulting matrix is typically larger for the nonlocal model due to the nonlocal interactions. Another difficulty arises when a Galerkin approach is used because the nonlocal integro-differential equation must be integrated over the spatial domain to obtain the weak formulation. This requires more complicated quadrature rules than when Galerkin methods are used to approximate standard PDE models, especially in the case of anomalous diffusion. In [4] the authors propose a coupled local-nonlocal model to help alleviate the computational costs of nonlocal problems for large-scale applications.

Radial basis functions (RBFs) are a class of functions which depend only on the distance to a fixed point so they are easily used on scattered grids and in higher dimensions. RBFs have their origins in techniques for performing function interpolation and were introduced in 1971 for topological interpolation using scattered data [5]. Since then they have become a powerful tool for multivariable interpolation problems [6]. Advantages of using RBFs for multivariable interpolation are their ease in using scattered data, their high rate of convergence and the fact that they are insensitive to the dimension of the space.

RBFs have been successfully used to solve PDEs from different standpoints. An RBF-based collocation method for elliptic problems was introduced in [7, 8] in 1990. Wendland [9] in 1999 used a Galerkin-RBF approach where the approximating functions and test functions were radial basis functions. In [10] the authors introduced an approach for using finite difference approximations on a scattered grid where RBFs were used to generate the finite difference stencils and in subsequent works [11, 12] use these RBF-generated stencils to solve the shallow water equations on the sphere.

In this work we use RBFs in a novel way to solve nonlocal diffusion problems. RBFs have been used to solve nonlocal problems when a Galerkin formulation is employed with RBFs as the approximating functions. See, for example [13, 14, 15]. Here we do not use a Galerkin method but instead approximate the time derivative by a standard backward difference formula (BDF) and approximate the nonlocal integral with an RBF-generated quadrature rule. This alleviates the difficulty previously described when using a Galerkin formulation. These quadrature rules are derived by extending the procedure of [10] for obtaining RBF-generated finite difference stencils on scattered grids. It is not feasible to use standard quadrature rules such as Newton-Cotes rules for approximating the nonlocal integral because of the singularity in the integrand; in addition, Gaussian quadrature rules are not practical when solving a nonlocal diffusion problem because each quadrature point is also a point where the time derivative must be approximated.

The RBF-generated quadrature rules are able to accurately approximate the singular nonlocal integral and as the number of quadrature points are increased near spectral accuracy can be obtained. However, conditioning of the matrix for deriving the quadrature rule becomes an issue when the grid spacing goes to zero; this conditioning issue is also present when RBFs are used for interpolation.

In § 2 we derive RBF-generated quadrature rules for a nonlocal integral in one dimension and present numerical results for approximating nonlocal integrals using these new quadrature rules. In § 3 we apply the RBF-generated quadrature rules to solve a nonlocal diffusion model in one dimension. In § 4 we extend the approach to two dimensions and indicate how it can be easily extended to three dimensions. Here we consider using both the tensor product of one dimensional rules as well as