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FINITE ELEMENT ERROR ANALYSIS OF A MANTLE CONVECTION MODEL

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This paper is dedicated to William J. Layton on the occasion of his 60-th birthday.

Abstract. A mantle convection model consisting of the stationary Stokes equations and a time-dependent convection-diffusion equation for the temperature is studied. The Stokes problem is discretized with a conforming inf-sup stable pair of finite element spaces and the temperature equation is stabilized with the SUPG method. Finite element error estimates are derived which show the dependency of the error of the solution of one problem on the error of the solution of the other equation. The dependency of the error bounds on the coefficients of the problem is monitored.

Key words. Mantel Convection, Stokes problem with variable viscosity, temperature problem with variable thermal convection, inf-sup stable finite elements, SUPG stabilization

1. Introduction

The process that occurs in the three-dimensional spherical shell between the crust and the core of the earth is called mantle convection. In this region, the magma moves very slowly. The movement is driven by the differences of the temperature at the hot core and the cool crust. Considering long time intervals, this movement is usually modeled with an incompressible viscous flow equation. Main features of this flow model are the high viscosity of order 10^{20} Pa s [9], the small value of the thermal diffusivity (order $\mathcal{O}(10^{-8}\ \mathrm{m^2/s})$ in [9]), and the dependency of the viscosity on other quantities, like the temperature. In turn, the temperature distribution is also driven by the movement of the magma, such that the modeling leads to a coupled problem. Simulations of mantle convection problems are quite challenging. One has to consider a three-dimensional problem in a very long time interval. With todays hardware and software capabilities, time intervals of almost 10^9 years are simulated [9], which results in performing many time steps. The resolution of important features, like plumes, requires to use adaptively refined grids. Massively parallel simulations with dynamic load balancing become necessary. The model (1) and (2) considered in this paper forms just the basic model. More advanced models use non-Newtonian fluids or they include a coupling to models for the behavior of the crust of the earth (solid material) to simulate the evolution of tectonic plates.

In this paper, the same model as in [27] is studied. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be bounded with polyhedral Lipschitz boundary $\partial\Omega$. Because of the large viscosity, the inertial term of the fundamental equations of fluid dynamics, the Navier–Stokes equations, can be neglected in mantle convection problems and thus, the equations reduce to the stationary incompressible Stokes equations. These equations with

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variable kinematic viscosity $\nu(\theta) > 0$ almost everywhere in Ω are given by

$$
\begin{array}{rcll}
-2\nabla \cdot (\nu(\theta)\,\mathbb{D}(\boldsymbol{u})) + \nabla p & = & \boldsymbol{f} - \boldsymbol{\beta}(\theta)\,\theta & \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u} & = & 0 & \text{in } \Omega, \\
\boldsymbol{u} & = & \boldsymbol{0} & \text{on } \partial\Omega,
\end{array} \tag{1}
$$

where $\boldsymbol{u}$ is the velocity field, the velocity deformation tensor $\mathbb{D}(\boldsymbol{u}) = \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)/2$ is the symmetric part of the gradient of $\boldsymbol{u}$, p is the fluid pressure, and $\boldsymbol{f}$ represents the body forces. Besides the dependency of the viscosity on the temperature θ, a further impact of the temperature θ is described by the function $\boldsymbol{\beta}$.

The equation for the temperature is time-dependent. It is a convection-diffusion equation with a nonlinear diffusion term since the thermal diffusivity κ depends on the temperature

$$
\begin{array}{rcll}
\partial_t\theta - \nabla \cdot (\kappa(\theta)\,\nabla\theta) + \boldsymbol{u} \cdot \nabla\theta & = & g & \text{in } (0,T] \times \Omega, \\
\theta & = & 0 & \text{in } (0,T] \times \partial\Omega, \\
\theta(0,\boldsymbol{x}) & = & \theta_0(\boldsymbol{x}) & \text{in } \Omega.
\end{array} \tag{2}
$$

Altogether, (1) and (2) form a coupled system of equations. For the sake of easy implementation and efficiency, algorithms for the numerical solution of (1), (2) may decouple these problems and linearizations might be applied. Two algorithms in this spirit are as follows. Given a partition of the time interval into time steps $0 = t_0 < t_1 < \ldots < t_N = T$:

Algorithm 1.1. *Nonlinear problem for the temperature.*

(1) *the initial condition θ_0 is given*
(2) *compute $(\boldsymbol{u}_0, p_0)$ with θ_0*
(3) *for $i = 1, \ldots, N$ do*
(4) *compute θ_i with $\boldsymbol{u}_{i-1}$ or some other extrapolation, solving a nonlinear problem*
(5) *compute $(\boldsymbol{u}_i, p_i)$ with θ_i*
(6) *end*

and

Algorithm 1.2. *Linear problem for the temperature.*

(1) *the initial condition θ_0 is given*
(2) *compute $(\boldsymbol{u}_0, p_0)$ with θ_0*
(3) *for $i = 1, \ldots, N$ do*
(4) *compute θ_i with θ_{i-1} and $\boldsymbol{u}_{i-1}$ or some other extrapolations solving a linear problem*
(5) *compute $(\boldsymbol{u}_i, p_i)$ with θ_i*
(6) *end*

The finite element error analysis presented in this paper will focus on the individual equations which are solved in steps 4 and 5.

Finite element analysis of (1), (2) are already presented in [26, 27]. In [26], the case of constant viscosity ($\nu = 1$) and thermal diffusivity is studied. In addition, the right-hand side of the Stokes equations depends linearly on the temperature in this paper. In both papers, the application of continuous piecewise linear (P_1) finite elements for all quantities is considered. This approach requires the use of a stabilization of the discretization of the Stokes equations since the used pair of finite element spaces for velocity and pressure does not satisfy a discrete inf-sup condition. In [26, 27], the method of Brezzi and Pitkäranta [3] is applied. The convection-diffusion equation (2) is usually convection-dominated. Also this feature requires