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RELIABLE AND EFFICIENT A POSTERIORI ERROR ESTIMATES OF DG METHODS FOR A SIMPLIFIED FRICTIONAL CONTACT PROBLEM

FEI WANG AND WEIMIN HAN

Abstract. A posteriori error estimators are studied for discontinuous Galerkin methods for solving a representative elliptic variational inequality of the second kind, known as a simplified frictional contact problem. The estimators are derived by relating the error of the variational inequality to that of a linear problem. Reliability and efficiency of the estimators are theoretically proved.

Key words. Elliptic variational inequality, discontinuous Galerkin method, a posteriori error estimators, reliability, efficiency.

1. Introduction

For more than three decades, adaptive finite element method (AFEM) has been an active research field in scientific computing. As an efficient numerical approach, it has been widely used for solving a variety of differential equations. Each loop of AFEM consists of four steps:

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}.$$

That is, in each loop, we first solve the problem on a mesh, then use a posteriori error estimators to mark those elements to be refined, and finally, refine the marked elements and get a new mesh. We can continue this process until the estimated error satisfies certain smallness criterion. The adaptive finite element method can achieve required accuracy with lower memory usage and less computation time.

A posteriori error estimators are computable quantities that provide the contribution of error on each element to the global error. They are used in adaptive algorithms to indicate which elements need to be refined or coarsened. To capture the true error as precisely as possible, they should have two properties: reliability and efficiency ([1, 4]). Hence, obtaining reliable and efficient error estimators is the key for successful adaptive algorithms. A variety of a posteriori error estimators have been proposed and analyzed in the literature. Many error estimators can be classified as residual type or recovery type ([1, 4]). Various residual quantities are used to capture lost information going from u to u_h, such as residual of the equation, residual from derivative discontinuity and so on. Another type of error estimators is gradient recovery, i.e., $||G(\nabla u_h) - \nabla u_h||$ is used to approximate $||\nabla u - \nabla u_h||$, where a recovery operator G is applied to the numerical solution u_h to rebuild the gradient of the true solution u. A posteriori error analysis has been well established for standard finite element methods for solving linear partial differential equations, and we refer the reader to [1, 4, 30].

Due to the inequality feature, it is more difficult to develop a posteriori error estimators for variational inequalities (VIs). However, numerous articles can be found on a posteriori error analysis of finite element methods for the obstacle

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problem, which is an elliptic variational inequality (EVI) of the first kind, e.g., [5, 15, 24, 26, 29, 34]. In [11], Braess demonstrated that a posteriori error estimators for finite element solutions of the obstacle problem can be derived by applying a posteriori error estimates for an associated linear elliptic problem. For VIs of the second kind, in [7, 8, 9, 10], the authors studied a posteriori error estimation and established a framework through the duality theory, but the efficiency was not completely proved. In [31], the ideas in [11] were extended to give a posteriori error analysis for VIs of the second kind. Moreover, a proof was provided for the efficiency of the error estimators.

In recent years, thanks to the flexibility in constructing feasible local shape function spaces and the advantage to capture non-smooth or oscillatory solutions effectively, discontinuous Galerkin (DG) methods have been widely used for solving various types of partial differential equations. When applying h-adaptive algorithm with standard finite element methods, one needs to choose the mesh refinement rule carefully to maintain mesh conformity and shape regularity. In particular, hanging nodes are not allowed without special treatment. For discontinuous Galerkin methods, the approximate functions are allowed to be discontinuous across the element boundaries, so general meshes with hanging nodes and elements of different shapes are acceptable. Advantages of DG methods include the flexibility of mesh-refinements and construction of local shape function spaces (hp-adaptivity), and the increase of locality in discretization, which is of particular interest for parallel computing. A historical account on the development of DG methods can be found in [16]. In [2, 3], Arnold et al. established a unified error analysis of nine DG methods for elliptic problems and several articles provided a posteriori error analysis of DG methods for elliptic problems (e.g. [6, 12, 14, 22, 25, 27]). Carstensen et al. presented a unified theory for a posteriori error analysis of DG methods in [13]. In [32], the authors extended ideas of the unified framework about DG methods for elliptic problems presented in [3] to solve the obstacle problem and a simplified frictional contact problem, and obtained a priori error estimates, which reach optimal order for linear elements. In [33], reliable a posteriori error estimators of the residual type were derived for DG methods for solving the obstacle problem, and efficiency of the estimators is theoretically explored and numerically confirmed. A posteriori error analysis of DG methods for the obstacle problem was also studied in [20].

A posteriori error analysis of DG methods has not been well-studied for variational inequalities of the second kind. In this paper, we explore this topic and study a posteriori error estimates of DG methods for solving a representative elliptic variational inequality of the second kind, namely, the simplified frictional contact problem. The estimators are derived by relating the error of the variational inequality to that of a corresponding linear problem. Furthermore, the reliability and efficiency of the estimators are theoretically proved. Even though we only consider the residual type error estimators in this paper, an analysis for gradient recovery type error estimation can be obtained by the techniques used in this paper and the standard argument of gradient recovery type error analysis for the second order elliptic equations.

The paper is organized as follows. In Section 2, we introduce the variational inequality problem and the DG schemes for solving it. Then we derive reliable a posteriori error estimators of residual type for the DG methods of the simplified frictional contact problem in Section 3. Finally, we prove efficiency of the proposed error estimators in Section 4.

2. The problem and DG formulations

Contact mechanics is a rich source of variational inequalities. Various frictional contact problems have been studied in the literature, see, e.g., [23, 21]. To simplify the notation, in this paper, we choose the simplified frictional contact problem as an example to develop a posteriori error analysis of DG methods in solving elliptic variational inequalities of the second kind. Similar a posteriori error analysis can be developed for DG methods in solving other elliptic variational inequalities of the second kind, including those arising in frictional contact problems.

2.1. The problem. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded domain with Lipschitz boundary Γ that is divided into two mutually disjoint parts, i.e., $\Gamma = \Gamma_1 \cup \Gamma_2$. Here Γ_1 is a relatively closed subset of Γ, and $\Gamma_2 = \Gamma\backslash\Gamma_1$. Given $f \in L^2(\Omega)$ and a constant $g > 0$, the simplified frictional contact problem is: find $u \in V = H^1_{\Gamma_1}(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1\}$ such that

$$a(u, v - u) + j(v) - j(u) \geq (f, v - u) \quad \forall\, v \in V, \tag{1}$$

where $(\cdot,\cdot)$ denotes the L^2 inner product in the domain Ω and

$$\begin{aligned} a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u\, v \, dx, \\ j(v) &= \int_{\Gamma_2} g\, |v| \, ds. \end{aligned}$$

The frictional contact problem is an example of elliptic variational inequalities of the second kind and it has a unique solution $u \in V$ ([18, 19]). Moreover, there exists a unique Lagrange multiplier $\lambda \in L^\infty(\Gamma_2)$ such that

$$a(u, v) + \int_{\Gamma_2} g\, \lambda\, v \, ds = (f, v) \quad \forall\, v \in V, \tag{2}$$

$$|\lambda| \leq 1, \quad \lambda\, u = |u| \quad \text{a.e. on } \Gamma_2. \tag{3}$$

From (2) and (3), we know that the solution u of (1) is the weak solution of the following boundary value problem

$$\begin{aligned} -\triangle u + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \\ \nabla u \cdot \boldsymbol{n} &= -g\lambda && \text{on } \Gamma_2, \end{aligned}$$

where $\boldsymbol{n}$ is the unit outward normal vector. For any $v \in V$, set

$$\ell(v) = \int_\Omega f\, v \, dx - \int_{\Gamma_2} g\, \lambda\, v \, ds.$$

Then we have by (2),

$$a(u, v) = \ell(v) \quad \forall\, v \in V. \tag{4}$$

Given a triangulation $\mathcal{T}_h$ of Ω, for a Lipschitz subdomain $\omega \subset \Omega$, we define

$$a_{\omega,h}(v, w) := \sum_{K \in \mathcal{T}_h} \int_{\omega \cap K} (\nabla v \cdot \nabla w + vw) \, dx$$

and

$$\|v\|_{1,\omega,h} := a_{\omega,h}(v, v)^{1/2}.$$

Then define

$$|\lambda|_{*,\gamma,h} := \sup \left\{ \int_\gamma g\, \lambda\, v \, ds : \; v \in H^1_h(\omega), \; \|v\|_{1,\omega,h} = 1 \right\}, \tag{5}$$

where γ is a measurable subset of $\partial\omega \cap \Gamma_2$ and $H_h^1(\omega) = \{v \in L^2(\omega) : v|_{K\cap\omega} \in H^1(K \cap \omega)\}$. If $\omega = \Omega$ and $\gamma = \Gamma_2$, the subscript ω and γ are omitted. We have

$$|\lambda|_{*,\gamma,h} = \|w\|_{1,\omega,h}, \tag{6}$$

where $w \in H_h^1(\omega)$ is the solution of the following auxiliary equation

$$a_{\omega,h}(w, v) = \int_\gamma g\ \lambda\ v\, ds \quad \forall\, v \in H_h^1(\omega). \tag{7}$$

The formula (6) can be proved by an argument similar to that found in [31].

2.2. Discontinuous Galerkin formulations. First, we introduce some notation. Let $\{\mathcal{T}_h\}$ be a family of triangulations of $\overline{\Omega}$ such that the minimal angle condition is satisfied. For a triangulation $\mathcal{T}_h$, let $\mathcal{E}_h$ be the set of all edges, $\mathcal{E}_h^i \subset \mathcal{E}_h$ the set of all interior edges, $\mathcal{E}_h^b := \mathcal{E}_h \backslash \mathcal{E}_h^i$ the set of all boundary edges, $\mathcal{E}_h^0 \subset \mathcal{E}_h$ the set of all edges not lying on Γ_2, $\mathcal{E}_h^1 := \mathcal{E}_h^0 \backslash \mathcal{E}_h^i$, $\mathcal{E}_h^2 := \mathcal{E}_h \backslash \mathcal{E}_h^0$, and define $\mathcal{E}(K)$ as the set of sides of K. Let $h_K = \mathrm{diam}(K)$ for $K \in \mathcal{T}_h$, $h_e = \mathrm{length}(e)$ for $e \in \mathcal{E}_h$, and $\mathcal{N}_h$ denote the set of nodes of $\mathcal{T}_h$. For any element $K \in \mathcal{T}_h$, define the patch set $\omega_K := \cup\{T \in \mathcal{T}_h, T \cap K \neq \emptyset\}$, and for any edge e shared by two elements K^+ and K^-, define $\omega_e := K^+ \cup K^-$. For a scalar-valued function v and a vector-valued function $\boldsymbol{q}$, let $v^i = v|_{\partial K^i}$, $\boldsymbol{q}^i = \boldsymbol{q}|_{\partial K^i}$, and $\boldsymbol{n}^i = \boldsymbol{n}|_{\partial K^i}$ be the unit normal vector external to ∂K^i with $i = \pm$. Define the average $\{\cdot\}$ and the jump $[\cdot]$ on an interior edge $e \in \mathcal{E}_h^i$ as follows:

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad [v] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-,$$
$$\{\boldsymbol{q}\} = \frac{1}{2}(\boldsymbol{q}^+ + \boldsymbol{q}^-), \quad [\boldsymbol{q}] = \boldsymbol{q}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}^- \cdot \boldsymbol{n}^-.$$

For a boundary edge $e \in \mathcal{E}_h^b$, we let

$$[v] = v\boldsymbol{n}, \quad \{q\} = q,$$

where $\boldsymbol{n}$ is the outward unit normal.

Let us define the following linear finite element spaces

$$V_h = \{v_h \in L^2(\Omega) :\ v_h|_K \in P_1(K)\ \forall\, K \in \mathcal{T}_h\},$$
$$W_h = \{\boldsymbol{w}_h \in [L^2(\Omega)]^2 :\ \boldsymbol{w}_h|_K \in [P_1(K)]^2\ \forall\, K \in \mathcal{T}_h\}.$$

We denote by ∇_h the broken gradient whose restriction on each element $K \in \mathcal{T}_h$ is equal to ∇. Define some seminorms and norms by the following relations:

$$\|v\|_K^2 = \int_K v^2 dx, \quad |v|_{1,K}^2 = \|\nabla v\|_K^2, \quad \|v\|_e^2 = \int_e v^2 ds,$$
$$\|v\|_{0,h}^2 = \sum_{K\in\mathcal{T}_h} \|v\|_K^2, \quad |v|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2, \quad \|v\|_{1,h}^2 = \|v\|_{0,h}^2 + |v|_{1,h}^2.$$

Throughout this paper, "$\lesssim \cdots$" stands for "$\le C \cdots$", where C denotes a generic positive constant dependent on the minimal angle condition but not on the element sizes, which may take different values at different occurrences.

Now, let us introduce the discontinuous Galerkin methods for solving the variational inequality (1). Here, we take the local DG method (LDG) as an example to show how to derive a posteriori error estimators of DG methods for solving the frictional contact problem (1). The derivation and analysis for the LDG method in this paper can be extended straightforward to other DG methods studied in [32].

The LDG method ([17]) for solving the frictional contact problem is to find $u_h \in V_h$ such that

$$B_h(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq (f, v_h - u_h) \quad \forall\, v_h \in V_h, \tag{8}$$

where

$$\begin{aligned} B_h(u,v) = &\int_\Omega (\nabla_h u \cdot \nabla_h v + u\,v)\,dx - \int_{\mathcal{E}_h^0} [u] \cdot \{\nabla_h v\}\,ds - \int_{\mathcal{E}_h^0} \{\nabla_h u\} \cdot [v]\,ds \\ &- \int_{\mathcal{E}_h^i} \boldsymbol{\beta} \cdot [u][\nabla_h v]\,ds - \int_{\mathcal{E}_h^i} [\nabla_h u] \boldsymbol{\beta} \cdot [v]\,ds \\ &+ (r_0([u]) + l(\boldsymbol{\beta} \cdot [u]), r_0([v]) + l(\boldsymbol{\beta} \cdot [v])) + \alpha_0^j(u,v). \end{aligned} \tag{9}$$

Here $\boldsymbol{\beta} \in [L^2(\mathcal{E}_h^i)]^2$ is a vector-valued function which is constant on each edge of $\mathcal{E}_h^i$, and $\alpha_0^j(u,v) = \int_{\mathcal{E}_h^0} \eta[u] \cdot [v]\,ds$ is the penalty term with the penalty weighting function $\eta : \mathcal{E}_h^0 \to \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \mathcal{E}_h^0$, η_e being a positive number on e. For any $\boldsymbol{w}_h \in W_h$, the lifting operators $r_0 : [L^2(\mathcal{E}_h^0)]^2 \to W_h$ and $l : L^2(\mathcal{E}_h^i) \to W_h$ are defined by

$$\int_\Omega r_0(\boldsymbol{q}) \cdot \boldsymbol{w}_h dx = -\int_{\mathcal{E}_h^0} \boldsymbol{q} \cdot \{\boldsymbol{w}_h\}\,ds, \qquad \int_\Omega l(v) \cdot \boldsymbol{w}_h dx = -\int_{\mathcal{E}_h^i} v\,[\boldsymbol{w}_h]\,ds \quad \forall\, \boldsymbol{w}_h \in W_h.$$

The bilinear form B_h is continuous and elliptic with respect to certain DG-norm, and therefore, in particular, the discrete problem has a unique solution $u_h \in V_h$ (see [3, 32]). Similar to the continuous problem, there exists a unique Lagrange multiplier $\lambda_h \in L^\infty(\Gamma_2)$ such that ([19])

$$B_h(u_h, v_h) + \int_{\Gamma_2} g\,\lambda_h v_h ds = (f, v_h) \quad \forall\, v_h \in V_h, \tag{10}$$

$$|\lambda_h| \leq 1, \quad \lambda_h u_h = |u_h| \quad \text{a.e. on } \Gamma_2. \tag{11}$$

Let

$$\ell_h(v) = (f, v) - \int_{\Gamma_2} g\,\lambda_h v ds.$$

Then (10) becomes

$$B_h(u_h, v_h) = \ell_h(v_h) \quad \forall\, v_h \in V_h. \tag{12}$$

For any $v \in V$, we know that $[u] = 0$ and $[v] = 0$ on $e \in \mathcal{E}_h^0$. Then we have from (2) that

$$B_h(u, v) = a(u, v) = \ell(v) \quad \forall\, v \in V \tag{13}$$

Obviously, u_h is also the finite element approximation of the solution $z \in V$ of the linear problem:

$$B_h(z, v) = \ell_h(v) \quad \forall\, v \in V, \tag{14}$$

which is the weak formulation of the boundary value problem

$$\begin{aligned} -\Delta z + z &= f \qquad \text{in } \Omega, \\ z &= 0 \qquad \text{on } \Gamma_1, \\ \frac{\partial z}{\partial n} &= -g\lambda_h \quad \text{on } \Gamma_2. \end{aligned} \tag{15}$$

2.3. A bridge between $u_h - u$ and $u_h - z$. In this subsection, we relate the error $e := u_h - u$ to $u_h - z$ through the inequality

$$\|e\|_{1,h} + |\lambda - \lambda_h|_{*,h} \lesssim \|u_h - z\|_{1,h} + \left(\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|[u_h]\|_e^2 \right)^{1/2}. \tag{16}$$

Then we use this relation to derive a posteriori error estimators for DG solutions of the simplified frictional contact problem by utilizing a posteriori error estimators of the related linear elliptic problem (15). Note that a similar approach can be applied to other elliptic variational inequalities of the second kind.

To derive the inequality (16), we first define a continuous piecewise linear function in $V_h \cap H^1_{\Gamma_1}(\Omega)$, whose value is close to the numerical solution. For any given $v_h \in V_h$, expressed as $v_h = \sum_{K \in \mathcal{T}_h} \sum_{j=1}^3 \alpha_K^{(j)} \phi_K^{(j)}$, where $\phi_K^{(j)}$, $1 \le j \le 3$, are the linear basis functions corresponding to the three vertices of K, we construct a function $\chi \in V_h \cap H^1_{\Gamma_1}(\Omega)$ as follows: At every interior node and the nodes on Γ_2 of the conforming mesh $\mathcal{T}_h$, the value of χ is set to be the average of the values of v_h computed from all the elements sharing that node, and $\chi = 0$ at the boundary nodes on Γ_1. For each $\nu \in \mathcal{N}_h$, let $\omega_\nu = \{K \in \mathcal{T}_h : \nu \in K\}$ and denote its cardinality by $|\omega_\nu|$, which is bounded by a constant depending only on the minimal angle condition of the mesh. To each node ν, the associated basis function $\phi^{(\nu)}$ is given by

$$\text{supp}\phi^{(\nu)} = \bigcup_{K \in \omega_\nu} K, \quad \phi^{(\nu)}|_K = \phi_K^{(j)} \text{ for } x_K^{(j)} = \nu.$$

Then we define $\chi \in V_h \cap H^1_{\Gamma_1}(\Omega)$ by

$$\chi = \sum_{\nu \in \mathcal{N}_h} \beta^{(\nu)} \phi^{(\nu)}, \quad \text{where } \beta^{(\nu)} = \frac{1}{|\omega_\nu|} \sum_{x_K^{(j)} = \nu} \alpha_K^{(j)} \quad \text{if } \nu \in \mathcal{N}_h \text{ and } \nu \notin \Gamma_1.$$

For nonconforming meshes, let $\mathcal{N}_h^0$ be the set of all hanging nodes. Then we construct χ from v_h the same way as in the conforming mesh case on all the nodes $\nu \in \mathcal{N}_h \backslash \mathcal{N}_h^0$. For an upper bound of the error $v_h - \chi$, we quote a result from [22] (which is Theorem 2.2 there for conforming meshes; the same result also holds for nonconforming meshes, which is Theorem 2.3 in [22]).

Lemma 2.1. *Let $\mathcal{T}_h$ be a conforming triangulation. Then for any $v_h \in V_h$, we can construct a continuous function $\chi \in V_h \cap H^1_{\Gamma_1}(\Omega)$ from v_h, such that*

$$\sum_{K \in \mathcal{T}_h} \|v_h - \chi\|_{i,K}^2 \le C \sum_{e \in \mathcal{E}_h^0} h_e^{1-2i} \|[v_h]\|_e^2, \quad i = 0, 1, \tag{17}$$

where the constant C is independent of mesh size and v_h, but it may depend on the lower bound of the minimal angle of the elements in $\mathcal{T}_h$.

Now, let us derive the inequality (16). From (13) and (14), for all $v \in V$, we have

$$B_h(u_h - u, v) = B_h(u_h - z, v) + B_h(z - u, v) = B_h(u_h - z, v) + \int_{\Gamma_2} g(\lambda - \lambda_h) v \, ds.$$

By the definition (9) and noticing $[v] = 0$ on each $e \in \mathcal{E}_h^0$, the above equation becomes

$$\begin{aligned}
&\widetilde{a}(e,v) - \int_{\mathcal{E}_h^0}[e]\cdot\{\nabla_h v\}\,ds - \int_{\mathcal{E}_h^i}\boldsymbol{\beta}\cdot[e][\nabla_h v]\,ds \\
=&\widetilde{a}(u_h - z, v) - \int_{\mathcal{E}_h^0}[u_h - z]\cdot\{\nabla_h v\}\,ds \\
&- \int_{\mathcal{E}_h^i}\boldsymbol{\beta}\cdot[u_h - z][\nabla_h v]\,ds + \int_{\Gamma_2} g(\lambda - \lambda_h)v\,ds,
\end{aligned}$$

where

$$\widetilde{a}(u,v) = \int_\Omega (\nabla_h u\cdot\nabla_h v + u\,v)\,dx.$$

Then,

$$\begin{aligned}
\widetilde{a}(e,v) =&\widetilde{a}(u_h - z, v) - \int_{\mathcal{E}_h^0}[u - z]\cdot\{\nabla_h v\}ds - \int_{\mathcal{E}_h^i}\boldsymbol{\beta}\cdot[u - z][\nabla_h v]\,ds \\
&+ \int_{\Gamma_2} g(\lambda - \lambda_h)v\,ds.
\end{aligned}$$

Note that $[u - z] = 0$ on each $e \in \mathcal{E}_h^0$. We have

$$\widetilde{a}(e,v) = \widetilde{a}(u_h - z, v) + \int_{\Gamma_2} g(\lambda - \lambda_h)v\,ds. \tag{18}$$

Let $\chi \in V_h \cap H^1_{\Gamma_1}(\Omega)$ be the function constructed from u_h, satisfying (17) for $v_h = u_h$. Taking $v := \chi - u = \chi - u_h + u_h - u$ in (18) and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|e\|_{1,h}^2 \le&\|u_h - z\|_{1,h}\left(\|\chi - u_h\|_{1,h} + \|e\|_{1,h}\right) + \|e\|_{1,h}\|\chi - u_h\|_{1,h} \\
&+ \int_{\Gamma_2} g(\lambda - \lambda_h)(\chi - u)\,ds \\
=&\|e\|_{1,h}\left(\|u_h - z\|_{1,h} + \|\chi - u_h\|_{1,h}\right) + \|u_h - z\|_{1,h}\|\chi - u_h\|_{1,h} \\
&+ \int_{\Gamma_2} g(\lambda - \lambda_h)(\chi - u)\,ds \\
\le&\frac{1}{2}\|e\|_{1,h}^2 + \frac{1}{2}\left(\|u_h - z\|_{1,h} + \|\chi - u_h\|_{1,h}\right)^2 + \|u_h - z\|_{1,h}\|\chi - u_h\|_{1,h} \\
&+ \int_{\Gamma_2} g(\lambda - \lambda_h)(\chi - u)\,ds.
\end{aligned}$$

Write

$$\int_{\Gamma_2} g(\lambda - \lambda_h)(\chi - u)\,ds = \int_{\Gamma_2} g(\lambda - \lambda_h)(u_h - u)\,ds + \int_{\Gamma_2} g(\lambda - \lambda_h)(\chi - u_h)\,ds.$$

Note that by (3) and (11), we have

$$\begin{aligned}
&\int_{\Gamma_2} g(\lambda - \lambda_h)(u_h - u)\,ds \\
=&\int_{\Gamma_2} g\,\lambda\,u_h ds - \int_{\Gamma_2} g\,\lambda\,u\,ds - \int_{\Gamma_2} g\,\lambda_h\,u_h\,ds + \int_{\Gamma_2} g\,\lambda_h u\,ds \\
\le&\int_{\Gamma_2} g\,|u_h|\,ds - \int_{\Gamma_2} g\,|u|\,ds - \int_{\Gamma_2} g\,|u_h|\,ds + \int_{\Gamma_2} g\,|u|\,ds = 0.
\end{aligned}$$

In addition,

$$\int_{\Gamma_2} g(\lambda-\lambda_h)(\chi-u_h)\,ds \le |\lambda-\lambda_h|_{*,h}\|\chi-u_h\|_{1,h}$$
$$\le \epsilon|\lambda-\lambda_h|_{*,h}^2+\frac{1}{4\epsilon}\|\chi-u_h\|_{1,h}^2.$$

Hence,

$$\|e\|_{1,h}^2 \lesssim \|u_h-z\|_{1,h}^2+\|\chi-u_h\|_{1,h}^2+\epsilon|\lambda-\lambda_h|_{*,h}^2.$$

Recalling (6), we have

$$|\lambda-\lambda_h|_{*,h}=\|u-z\|_{1,h}\le\|e\|_{1,h}+\|u_h-z\|_{1,h}.$$

Then, we obtain the following result

$$\|e\|_{1,h}+|\lambda-\lambda_h|_{*,h}\lesssim\|u_h-z\|_{1,h}+\|\chi-u_h\|_{1,h}.$$

Using (17) with $i=1$ to bound $\|\chi-u_h\|_{1,h}$, the above inequality can be rewritten as

$$\|e\|_{1,h}+|\lambda-\lambda_h|_{*,h}\lesssim\|u_h-z\|_{1,h}+\left(\sum_{e\in\mathcal{E}_h^0}h_e^{-1}\|[u_h]\|_e^2\right)^{1/2}. \tag{19}$$

The relation (19) serves as a starting point for derivation of reliable and efficient error estimators of DG methods for a frictional contact problem. In this paper, we focus on the derivation and analysis of residual type error estimators derived from the inequality (19). A similar approach can also be applied to recovery type error estimators.

3. Reliable residual-type estimators

Now we derive a posteriori error estimators of DG methods for solving the simplified frictional contact problem. The detailed derivation and analysis of a posteriori error estimators is given for the LDG method [17]. For other DG methods discussed in [32], similar results can be obtained by similar arguments.

Given interior residuals and edge-based jumps as follows,

$$R_K:=\Delta u_h-u_h+f \quad \text{for each } K\in\mathcal{T}_h,$$
$$R_e:=\begin{cases}[\nabla_h u_h] & \text{if } e\in\mathcal{E}_h^i,\\ \nabla_h u_h\cdot n+g\lambda_h & \text{if } e\in\mathcal{E}_h^2,\end{cases}$$

we define local estimators as

$$\eta_K:=\left(h_K^2\|R_K\|_K^2+\frac{1}{2}\sum_{e\in\partial K\cap\mathcal{E}_h^i}h_e\|R_e\|_e^2+\sum_{e\in\partial K\cap\mathcal{E}_h^2}h_e\|R_e\|_e^2\right)^{1/2}, \tag{20}$$

$$\eta_{\partial K}:=\left(\frac{1}{2}\sum_{e\in\partial K\cap\mathcal{E}_h^i}h_e^{-1}\|[u_h]\|_e^2+\sum_{e\in\partial K\cap\mathcal{E}_h^1}h_e^{-1}\|[u_h]\|_e^2\right)^{1/2}. \tag{21}$$

Applying Corollary 3.3, proved at the end of this section, to bound $\|u_h-z\|_{1,h}$ in (19), we have the following theorem.

Theorem 3.1. *Let $u\in H^2(\Omega)$ and u_h solve (1) and (8) respectively. Then we have*

$$\|u-u_h\|_{1,h}+|\lambda-\lambda_h|_{*,h}\lesssim\left(\sum_{K\in\mathcal{T}_h}\eta_K^2+\sum_{K\in\mathcal{T}_h}\eta_{\partial K}^2\right)^{1/2}. \tag{22}$$

Now, let us show how to derive the result in Corollary 3.3 to bound the term $\|u_h - z\|_{1,h}$. We recall one result in [13]. Note that the a posteriori error analysis in [13] was only for the Poisson problem with homogenous Dirichlet boundary condition, but it is easy to extend the result to general elliptic problems with Neumann boundary conditions. For the second-order elliptic problem

$$-\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2,$$

rewrite it as a first order system

$$p = \nabla u, \; -\nabla \cdot p + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2. \tag{23}$$

Then the DG formulation for the problem is

$$\int_\Omega p_h \cdot \tau_h dx = -\int_\Omega u_h \nabla_h \cdot \tau_h dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \hat{u_h} n_K \cdot \tau_h ds \qquad \forall \tau_h \in W_h, \tag{24}$$

$$\int_\Omega (p_h \cdot \nabla_h v_h + u_h v_h)\, dx = \int_\Omega f\, v_h dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \hat{p_h} \cdot n_K v_h ds \quad \forall v_h \in V_h, \tag{25}$$

where $\hat{u_h}$ and $\hat{p_h}$ are numerical fluxes. Different choice of the numerical fluxes leads to different DG method. The following result (see [13]) holds for the LDG method and other methods discussed in [3].

Theorem 3.2. *Assume $u \in H^1_{\Gamma_1}(\Omega)$ and $p \in W := [L^2(\Omega)]^2$ are the solution of the problem (23), and $u_h \in V_h$ and $p_h \in W_h$ are the solution of the problem (24)–(25). Then,*

$$\|p - p_h\| \le C\left(\eta_* + \zeta_*\right),$$

where

$$\eta_*^2 := \sum_{K\in\mathcal{T}_h} h_K^2 \|\text{div} p_h - u_h + f\|_K^2 + \sum_{e\in\mathcal{E}_h^i} h_e \|[p_h]\|_e^2 + \sum_{e\in\mathcal{E}_h^2} h_e \|p_h \cdot n - g\|_e^2,$$

$$\zeta_*^2 := \sum_{e\in\mathcal{E}_h^0} h_e^{-1} \|[u_h]\|_e^2$$

and C is a mesh-size independent constant which depends only on the domain Ω and the minimal angle condition.

From the relation between p_h and u_h ([3, 13]), we deduce the following result.

Corollary 3.3. *With the same notation as in Theorem 3.2, we have*

$$\|\nabla u - \nabla_h u_h\| \le C(\eta + \zeta_*),$$

where

$$\eta^2 := \sum_{K\in\mathcal{T}_h} h_K^2 \|\Delta u_h - u_h + f\|_K^2 + \sum_{e\in\mathcal{E}_h^i} h_e \|[\nabla_h u_h]\|_e^2 + \sum_{e\in\mathcal{E}_h^2} h_e \|\nabla_h u_h \cdot n - g\|_e^2.$$

Proof. By [28, Lemma 7.2], for any $v_h \in V_h$,

$$\|r_0([v_h])\|^2 \le C \sum_{e\in\mathcal{E}_h^0} h_e^{-1} \|[v_h]\|_e^2, \qquad \|l(\boldsymbol{\beta}\cdot[v_h])\|^2 \le C \sum_{e\in\mathcal{E}_h^i} h_e^{-1} \|[v_h]\|_e^2.$$

From [3, (3.9)], we know that

$$p_h = \nabla_h u_h - r_0([\hat{u_h} - u_h]) - l(\{\hat{u_h} - u_h\}).$$

Then

$$\begin{aligned}\|\nabla u-\nabla_h u_h\| &\le \|\nabla u-p_h\|+\|p_h-\nabla_h u_h\|\\ &\le C\left(\eta_*+\zeta_*\right)+\|r_0([\hat{u_h}-u_h])\|+\|l(\{\hat{u_h}-u_h\})\|.\end{aligned}$$

From the choices of numerical fluxes $\hat{u_h}$ in Table 3.1 of [3], we have

$$[\hat{u_h}-u_h]=-[u_h] \text{ or } 0, \qquad \{\hat{u_h}-u_h\}=-\boldsymbol{\beta}\cdot[u_h] \text{ or } 0.$$

So

$$\|r_0([\hat{u_h}-u_h])\|\le C\sum_{e\in\mathcal{E}_h^0} h_e^{-1}\|[u_h]\|_e^2, \qquad \|l(\{\hat{u_h}-u_h\})\|\le C\sum_{e\in\mathcal{E}_h^i} h_e^{-1}\|[u_h]\|_e^2,$$

which implies

$$\|p_h-\nabla_h u_h\|\le\zeta_* \quad \text{and} \quad \|\nabla u-\nabla_h u_h\|\le C\left(\eta_*+\zeta_*\right).$$

Finally, by the inverse inequality and trace inequality, we get

$$\begin{aligned}\eta_*^2 =&\sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div} p_h-u_h+f\|_K^2+\sum_{e\in\mathcal{E}_h^i} h_e\|[p_h]\|_e^2+\sum_{e\in\mathcal{E}_h^2} h_e\|p_h\cdot n-g\|_e^2\\ \le& 2\Big(\eta^2+\sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}(p_h-\nabla u_h)\|_K^2+\sum_{e\in\mathcal{E}_h^i} h_e\|[p_h-\nabla_h u_h]\|_e^2\\ &+\sum_{e\in\mathcal{E}_h^2} h_e\|(p_h-\nabla_h u_h)\cdot n\|_e^2\Big)\\ \le& 2\eta^2+2\sum_{K\in\mathcal{T}_h} h_K^2\|\mathrm{div}(p_h-\nabla u_h)\|_K^2+C\Big(\sum_{K\in\mathcal{T}_h}\|p_h-\nabla u_h\|_K^2\\ &+\sum_{K\in\mathcal{T}_h} h_K^2|p_h-\nabla u_h|_{1,K}^2\Big)\\ \le& 2\eta^2+C\sum_{K\in\mathcal{T}_h}\|p_h-\nabla u_h\|_K^2=2\eta^2+C\|p_h-\nabla_h u_h\|^2\le 2\eta^2+C\zeta_*^2.\end{aligned}$$

Therefore, $\eta_*\le C\left(\eta+\zeta_*\right)$ and the result is proved. □

4. Efficiency of the estimators

Now we present lower bounds of the estimators. We follow the standard argument to derive lower bounds of residual error estimators for elliptic problems, see [1, pp. 28–31]. First, we introduce the bubble functions. Let $K\in\mathcal{T}_h$, and let λ_1, λ_2 and λ_3 be the barycentric coordinates on K. Then the interior bubble function φ_K is defined by

$$\varphi_K=27\lambda_1\lambda_2\lambda_3$$

and the three edge bubble functions are given by

$$\tau_1=4\lambda_2\lambda_3, \quad \tau_2=4\lambda_1\lambda_3, \quad \tau_3=4\lambda_1\lambda_2.$$

We list properties of bubble functions stated in Theorems 2.2 and 2.3 of [1] in the form of a lemma.

Lemma 4.1. *For each $K\in\mathcal{T}_h$, $e\subset\partial K$, let φ_K and τ_e be the corresponding interior and edge bubble functions. Let $P(K)\subset H^1(K)$ and $P(e)\subset H^1(e)$ be finite-dimensional spaces of functions defined on K or e. Then there exists a constant C*

independent of h_K such that for all $v \in P(K)$,

$$C^{-1}\|v\|_K^2 \le \int_K \varphi_K v^2\,dx \le C\|v\|_K^2,$$
$$C^{-1}\|v\|_K \le \|\varphi_K v\|_K + h_K|\varphi_K v|_{1,K} \le C\|v\|_K,$$
$$C^{-1}\|v\|_e^2 \le \int_e \tau_e v^2\,ds \le C\|v\|_e^2,$$
$$h_K^{-1/2}\|\tau_e v\|_K + h_K^{1/2}|\tau_e v|_{1,K} \le C\|v\|_e.$$

Denote

$$a_K(u,v) = \int_K (\nabla u\cdot\nabla v + uv)\,dx.$$

Then for $u, v \in H^1(\Omega)$,

$$a(u,v) = \sum_{K\in\mathcal{T}_h} a_K(u,v).$$

For all $v \in H^1_{\Gamma_1}(\Omega)$, noting that $[v] = 0$ and $[u-z] = 0$ on $e \in \mathcal{E}_h^0$, we have

$$\sum_{K\in\mathcal{T}_h} a_K(e,v) = \sum_{K\in\mathcal{T}_h} a_K(u_h - z, v) + a(z-u,v)$$
$$= \sum_{K\in\mathcal{T}_h} a_K(u_h - z, v) + \int_{\Gamma_2} g(\lambda - \lambda_h)v\,ds.$$

Now,

$$\sum_{K\in\mathcal{T}_h} a_K(u_h - z, v) = \sum_{K\in\mathcal{T}_h}\int_K \big(\nabla(u_h-z)\cdot\nabla v + (u_h - z)v\big)\,dx$$
$$= \sum_{K\in\mathcal{T}_h}\int_K \big(-\Delta(u_h-z) + u_h - z\big)v\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\nabla(u_h - z)\cdot n_K v\,ds.$$

Hence,

$$\sum_{K\in\mathcal{T}_h} a_K(e,v) = \sum_{K\in\mathcal{T}_h}\int_K(-\Delta u_h + u_h - f)v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e[\nabla u_h]\cdot v\,ds$$
$$+ \sum_{e\in\mathcal{E}_h^2}\int_e(\nabla u_h\cdot n + g\lambda_h)v\,ds + \int_{\mathcal{E}_h^2} g(\lambda-\lambda_h)v\,ds. \tag{26}$$

For each $K \in \mathcal{T}_h$, φ_K and τ_e are respectively the interior and edge bubble functions on K or $e \in \mathcal{E}_h^i \cup \mathcal{E}_h^2$. $\bar R_K$ is an approximation to the interior residual R_K from a suitable finite-dimensional subspace. In (26), choose $v = \bar R_K\varphi_K$ on element K. We know φ_K vanishes on the boundary of K by its definition, so v can be extended to be zero on the rest of domain as a continuous function. Therefore, we get

$$a_K(e, \bar R_K\varphi_K) = \int_K R_K\bar R_K\varphi_K\,dx.$$

Then

$$\int_K \bar R_K^2\varphi_K\,dx = \int_K \bar R_K(\bar R_K - R_K)\varphi_K\,dx + a_K(e,\bar R_K\varphi_K).$$

Applying the Cauchy-Schwarz inequality and Lemma 4.1, we obtain

$$\int_K \bar R_K(\bar R_K - R_K)\varphi_K\,dx \le \|\bar R_K\varphi_K\|_K\|\bar R_K - R_K\|_K \lesssim \|\bar R_K\|_K\|\bar R_K - R_K\|_K,$$
$$a_K(e,\bar R_K\varphi_K) \le \|e\|_{1,K}\|\bar R_K\varphi_K\|_{1,K} \lesssim h_K^{-1}\|e\|_{1,K}\|\bar R_K\|_K.$$

Use Lemma 4.1 again,

$$\|\bar{R}_K\|_K^2 \lesssim \int_K \bar{R}_K^2 \varphi_K dx.$$

Combining the above relations, we obtain

$$\|\bar{R}_K\|_K \lesssim \|\bar{R}_K - R_K\|_K + h_K^{-1}\|e\|_{1,K}.$$

Finally, by the triangle inequality $\|R_K\|_K \le \|R_K - \bar{R}_K\|_K + \|\bar{R}_K\|_K$, we get

$$\|R_K\|_K \lesssim \|\bar{R}_K - R_K\|_K + h_K^{-1}\|e\|_{1,K}.$$

Now choose the finite-dimensional subspace from which the $\bar{R}_K$ come as the function space spanned by the local nodal basis $\phi_K^{(i)}$ with $i = 1, 2, 3$. Then, $\|\bar{R}_K - R_K\|_K$ reduces to $\|f - \overline{f}\|_K$ where we take

$$\overline{f} = \sum_{i=1}^{3} f^i \phi_K^{(i)} \qquad \text{with} \quad f^i = (f, \phi_K^{(i)})_K/(1, \phi_K^{(i)})_K. \tag{27}$$

For $e \in \mathcal{E}_h^2$, we obtain

$$a_{\omega_e}(u_h - u, \overline{R}_e\tau_e) = \int_{\omega_e} R_K \overline{R}_e \tau_e dx + \int_e R_e \overline{R}_e \tau_e ds + \int_e g(\lambda - \lambda_h)\overline{R}_e \tau_e ds$$

and therefore

$$\begin{aligned}\int_e \overline{R}_e^2 \tau_e ds = &\int_e \overline{R}_e(\overline{R}_e - R_e)\tau_e ds + a_{\omega_e}(u_h - u, \overline{R}_e\tau_e)\\ &- \int_{\omega_e} R_K \overline{R}_e \tau_e dx - \int_e g(\lambda - \lambda_h)\overline{R}_e\tau_e ds.\end{aligned}$$

From Lemma 4.1, we estimate the terms in above relation as

$$\begin{aligned}
C^{-1}\|\overline{R}_e\|_e^2 &\le \int_e \overline{R}_e^2 \tau_e\, ds,\\
\int_e \overline{R}_e(\overline{R}_e - R_e)\tau_e\, ds &\le \|\overline{R}_e\tau_e\|_e \|\overline{R}_e - R_e\|_e \le C\|\overline{R}_e\|_e\|\overline{R}_e - R_e\|_e,\\
a_{\omega_e}(u_h - u, \overline{R}_e\tau_e) &\le \|u_h - u\|_{1,\omega_e}\|\overline{R}_e\tau_e\|_{1,\omega_e} \le C h_e^{-1/2}\|u_h - u\|_{1,\omega_e}\|\overline{R}_e\|_e,\\
\int_{\omega_e} R_K\overline{R}_e\tau_e\, dx &\le \|R_K\|_{\omega_e}\|\overline{R}_e\tau_e\|_{\omega_e} \le C h_e^{1/2}\|R_K\|_{\omega_e}\|\overline{R}_e\|_e,\\
\int_e g(\lambda - \lambda_h)\overline{R}_e\tau_e\, ds &\le |\lambda - \lambda_h|_{*,e}\|\overline{R}_e\tau_e\|_{1,\omega_e} \le C h_e^{-1/2}|\lambda - \lambda_h|_{*,e}\|\overline{R}_e\|_e.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|R_e\|_e \le \|\overline{R}_e\|_e + \|R_e - \overline{R}_e\|_e\\
\le &C\big(h_e^{-1/2}\|u_h - u\|_{1,\omega_e} + h_e^{-1/2}|\lambda - \lambda_h|_{*,e} + h_e^{1/2}\|R_K - \overline{R}_K\|_{\omega_e} + \|R_e - \overline{R}_e\|_e\big).
\end{aligned}$$

For $e \in \mathcal{E}_h^i$, let $\overline{R}_e$ be an approximation to the jump R_e from a suitable finite-dimensional space and let $v = \overline{R}_e\tau_e$ in (26). By a similar argument, we have

$$\|R_e\|_e \le C\big(h_e^{-1/2}\|u_h - u\|_{1,\omega_e} + h_e^{1/2}\|R_K - \overline{R}_K\|_{\omega_e} + \|R_e - \overline{R}_e\|_e\big).$$

Note that $\Delta u_h + u_h$ in K and $\partial u_h/\partial n_e$ on e are polynomials. Hence, the terms $\|R_K - \overline{R}_K\|_K$ and $\|R_e - \overline{R}_e\|_e$ can be replaced by $\|f - \overline{f}\|_K$ and $\|\lambda_h - \overline{\lambda}_h\|_e$, with discontinuous piecewise polynomial approximations $\overline{\lambda}_h$. Then we obtain the efficiency bound of the local error indicator η_K.

Theorem 4.2. *Let u and u_h be the solutions of* (1) *and* (8)*, respectively, and η_K be the estimator* (20)*. Then*

$$\eta_K \le C\Big(|u-u_h|_{\omega_K} + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_2} |\lambda-\lambda_h|_{*,e} + h_K\|f-f_h\|_{\omega_K} + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_2} h_e\|\lambda_h-\overline{\lambda}_h\|_e\Big), \tag{28}$$

where the constant C is dependent on the angle condition and independent of h_K.

5. Summary

In this paper, we provide a posteriori error analysis of DG methods for a representative elliptic variational inequality of the second kind, the simplified frictional contact problem. By relating the error of the variational inequality to the error of a corresponding linear problem, we derive residual-type error estimators, which are theoretically proved to be reliable and efficient. Note that we consider only the residual type error estimators in this paper, but the analysis for gradient recovery type error estimation can be obtained by the techniques used in this paper and the standard argument of gradient recovery type error analysis for second order elliptic boundary value problems.

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School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China
E-mail: `feiwang.xjtu@xjtu.edu.cn`
URL: `http://gr.xjtu.edu.cn/web/fei/`

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China, and Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
E-mail: `weimin-han@uiowa.edu`
URL: `http://homepage.divms.uiowa.edu/~whan/`