INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 1, Pages 97–115

PATTERN FORMATION IN ROSENZWEIG–MACARTHUR MODEL WITH PREY–TAXIS

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Abstract. In this paper we study the existence and stability of nonconstant positive steady states to a reaction–advection–diffusion system with Rosenzweig–MacArthur kinetics. This system can be used to model the spatial–temporal distributions of predator and prey species . We investigate the effect of prey–taxis on the formation of nonconstant positive steady states in 1D. Stability and instability of these nonconstant steady states are also obtained. We also perform some numerical studies to support the theoretical findings. It is also shown that the Rosenzweig–MacArthur prey–taxis model admits very rich and complicated spatial–temporal dynamics.

Key words. Predator–prey, prey–taxis, steady state, stability analysis.

1. Introduction and preliminary results

Recently there has been great interest in the mathematical modeling and analysis of spatial–temporal population distributions of interacting species in biology and ecology. In the world of living things, one of the characteristic features of organisms is their ability to sense the stimulating signals in the environment and adjust movements accordingly. In predator–prey interactions, prey–taxis is the directed movement of predator species along the gradient of high prey population density. Prey–taxis is called positive or prey–attractive if predators move towards and forage the high density prey, while it is called negative or prey–repulsive if predators move against and retreat from preys' habitat. This is very similar as chemotaxis in which cellular bacteria move in response to chemical stimulus in their environment [11, 13, 15, 21].

We consider a reaction–advection–diffusion system of $u = u(x,t)$ and $v = v(x,t)$ in the following form

$$(1)\qquad \begin{cases} u_t = d_1\Delta u + u(a(1-\frac{u}{h}) - \frac{bv}{u+c}), & x\in\Omega, t>0,\\ v_t = \nabla\cdot(d_2\nabla v - \chi\phi(u,v)\nabla u) + v(\frac{eu}{u+c} - d), & x\in\Omega, t>0,\\ \partial_{\mathbf{n}}u = \partial_{\mathbf{n}}v = 0 & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 1$, is a bounded domain with smooth boundary $\partial\Omega$; $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ and $\nabla = (\frac{\partial}{\partial x_1},...\frac{\partial}{\partial x_i},...,\frac{\partial}{\partial x_N})$. d_1, d_2, a,b,c,d,e and h are all positive constants. ϕ is assumed to be a smooth function of u and v, and $\phi(0,v)=0$ for all $v>0$, which describes the biologically realistic situation that there is no prey–taxis in the absence of prey species; $\partial_{\mathbf{n}}$ denotes the unit outer normal derivative on the boundary. System (1) is a prey–taxis model, where u and v denote population densities of prey and predator species at space–time location (x,t) respectively. The movement of prey u is purely diffusive, while that of predator v is both diffusive and advective. d_1 and d_2 are random dispersal rates of prey and predator. χ measures the strength of prey–taxis. For example, if $\phi(u,v)>0$ for all $u,v>0$,

Received by the editors December 7, 2017 and, in revised form, April 1, 2018.
2000 *Mathematics Subject Classification.* 92C17, 35B32, 35B35, 35B36, 35J47, 35K20, 37K45, 37K50.

then the prey–taxis is positive if $\chi > 0$ and it is negative if $\chi < 0$. Here the potential function ϕ reflects the intensity of such directed dispersal with respect to the variation of both predator and prey densities. We would like to remark the opposite case $\phi(u,v) < 0$ can be used to model volume–filling effect in predators or group–defense in preys. See our discussions in the last paragraph of this section. It is the goal of this paper to study the existence and stability of nonconstant positive solutions to the stationary system of (1).

The reaction system in (1) or its ODE system is referred to as the Rosenzweig–MacArthur model [34, 43]

$$\begin{cases} u_t = u(a(1-\frac{u}{h}) - \frac{bv}{u+c}), & t>0, \\ v_t = v(\frac{eu}{u+c} - d), & t>0, \\ u(0)=u_0, v(0)=v_0, \end{cases}$$

which has been widely applied in real–life ecology [36]. This model is also known as the Lotka–Volterra equations with a Holling type II predator functional response or the Gause model. See [34, 35] and Chap. 4 in [9] for works on this system and similar modified Lotka–Volterra equations. Here a is the intrinsic growth rate of the predator, h is the environment carrying capacity, b and e are the interaction rates for the two species, and d is the intrinsic death rate of the predator. c measures the saturation effect on the predator growth due to the consumption of prey at a unit number.

The Rosenzweig–MacArthur ODE model has been investigated by various authors. It is easy to see that it has three equilibrium points $(0,0)$, $(h,0)$ and

$$(\bar{u},\bar{v}) = \Big(\frac{cd}{e-d}, \frac{ace\big(h(e-d)-cd\big)}{bh(e-d)^2}\Big), \tag{2}$$

where $(\bar{u},\bar{v})$ is positive if and only if $0 < cd < h(e-d)$. For the sake of our mathematical analysis, throughout this paper we make the following assumptions

$$d < e, \frac{cd}{e-d} < h < \frac{c(d+e)}{e-d}. \tag{3}$$

By the standard ODE stability analysis, see [9, 17] e.g., the first two equilibrium points $(0,0)$ and $(h,0)$, corresponding to the extinction of species and predators respectively, are both saddle points. $(\bar{u},\bar{v})$ corresponds to the coexistence of both species and it undergoes a Hopf bifurcation as e increases . Moreover, according to [2], this equilibrium loses its stability to a small amplitude periodic orbit which is unique hence stable. The relaxation oscillator profile of the unique limit cycle of the ODE system is discussed in [16]. See [57] and the references therein for more discussions. We also want to point out that Rinaldi *et al.* [42] studied this model with time–periodically varying parameters and identified six elementary seasonality mechanisms through complete bifurcation diagrams.

In the absence of prey–taxis, system (1) reads

$$\begin{cases} u_t = d_1\Delta u + u(a(1-\frac{u}{h}) - \frac{bv}{u+c}), & x\in\Omega, t>0, \\ v_t = d_2\Delta v + v(\frac{eu}{u+c} - d), & x\in\Omega, t>0, \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0 & x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{4}$$

and this model has also been extensively studied by various authors. Choosing $d_2 = 0$ and scaling the rest parameters in (4), Dunbar [7] obtained periodic traveling wave train and traveling front solutions for this diffusive predator–prey system. His analysis also shows the existence of periodic orbits, heteroclinic orbits and a