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# A NEW *A POSTERIORI* ERROR ESTIMATE FOR THE INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD

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**Abstract.** In this paper, we develop the adaptive interior penalty discontinuous Galerkin method based on a new *a posteriori* error estimate for the second-order elliptic boundary-value problems. The new *a posteriori* error estimate is motivated from the smoothing iteration of the $m$-time Gauss-Seidel iterative method, and it is used to construct the adaptive finite element method. The efficiency and robustness of the proposed adaptive method is demonstrated by extensive numerical experiments.

**Key words.** Interior penalty discontinuous Galerkin method, *a posteriori* error estimate, adaptive finite element methods, Gauss-Seidel iterative method.

## 1. Introduction.

The finite element method (**FEM**) is one of the most important computational tools in the field of science and engineering. The discontinuous Galerkin (**DG**) method is an innovation, improvement and development of the **FEM**. Since the **DG** method has advantages in parallel computing of the adaptive finite element method (**AFEM**), it has been favored by researchers and gradually become one of the important numerical methods in solving all kinds of partial differential equations. The **DG** method and theory have achieved fruitful results in recent years. The interior penalty discontinuous Galerkin (**IPDG**) method belongs to the family of symmetric **DG** methods. It has locally conservation, stability and high-order accuracy which can easily handle complex geometries, irregular meshes with hanging nodes and approximations with polynomials of different degrees in different elements. In 1982, Arnold [2] introduced the first **IPDG** method for heat equations, and now this method has already been applied in engineering increasingly, especially in the computational electromagnetics. For example, it's used in solving Maxwell's equations in cold plasma and dispersive media [17, 18], indefinite time-harmonic Maxwell's equations [16], compressible Navier-Stokes equations [15] and Helmholtz equations with spatially varying wavenumber [11], etc. The construction of a reliable and efficient error estimator is essential to the success of adaptive algorithms. In general, engineering calculations are mainly based on the first or second order finite element methods, and engineers hope to have a better precision for these low-order finite elements. However, most of the time the exact solutions are unknown, and the errors can not be calculated directly. In such cases, many researchers pay their attention to *a posteriori* error estimates, and the real errors can be better approximated by post-processing techniques with the obtained finite element solutions. The *a posteriori* error estimate and the **AFEM** were first introduced by [4]. Since the late 1980s, the residual type *a posteriori* error estimate [8], the recovery type *a posteriori* error estimate [26], the *a posteriori* error estimate based on the hierarchic basis [6, 7], a new *a posteriori* error estimate for **AFEM**

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[19] and a series of works on the *a posteriori* error estimation [1, 5, 24, 25] have been put forward one after another.

With the rapid development of the **DG** theory, the *a posteriori* error estimation theory based on **DG** methods arises. There are a large number of numerical and theoretical literatures on **AFEM**, and many scholars also put forward to different kinds of *a posteriori* error estimates for various problems. Since 2000, much research work on the *a posteriori* error estimation has been developed including the residual *a posteriori* error estimate for the **DG** approximation of second-order elliptic problems [20], and the *a posteriori* error estimations and mesh adaptivity strategies for **DG** methods applied to diffusion problems [22]. The **IPDG** method has many good properties, which render it ideal to be used with the local mesh refinement and the independent selection of the polynomial degree in each element. Those distinct advantages make the computation of finite elements more efficient and flexible. At the same time, owing to the degrees of freedom for each unit are less than other **DG** methods, the **IPDG** formulation is relatively easier to be paralleled. In this paper we aim to propose a new *a posteriori* error estimate for the **IPDG** method by handling the numerical solution in a simple way. It is shown to be a simple and efficient way to improve the approximation accuracy of the numerical solution with less computational cost.

In this paper, we consider the **IPDG** method for a second-order elliptic boundary-value problem. We present a new *a posteriori* error estimator with $m$-time Gauss-Seidel iterations in an energy norm (cf. G-norm below) and use it to construct the **AFEM**. In particular, on the current triangulation $\mathcal{T}_h$, we solve the equation to obtain the **DG** solution $u_h$, then globally refine $\mathcal{T}_h$ to obtain an auxiliary mesh $\mathcal{T}_{h/2}$. On the fine mesh, we use a simple smoother such as the Gauss-Seidel iterations with $u_h$ as the initial value. After $m$-time iterations, we obtain an approximation $u_{h/2,m}$ of the **DG** solution $u_{h/2}$ on the fine mesh $\mathcal{T}_{h/2}$. We then take $\|u_{h/2,m} - u_h\|_G$ as the *a posteriori* error estimator to guide the mesh refinement on $\mathcal{T}_h$. In practice, it only needs a small number of smoothing steps to obtain an efficient *a posteriori* error estimator, hence the computational cost is relatively small.

The rest of the paper is organized as follows. In section 2, we propose a new *a posteriori* error estimator and then describe the **AFEM** algorithm with it for second-order elliptic boundary-value problem. We present some numerical results to show the efficiency of the new *a posteriori* error estimator and the performance of the corresponding **AFEM** algorithm in section 3.

## 2. The New *A Posteriori* Error Estimate.

**2.1. The Interior Penalty Discontinuous Galerkin Method.** In this work, we consider the following second-order elliptic boundary-value problem:

$$-\nabla \cdot (a\nabla u) + bu = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^2$, $\partial\Omega$ is the Lipschitz boundary. For the sake of simplicity and easy presentation, we consider the homogeneous boundary condition, i.e., $g = 0$ and $\Omega$ is assumed to be a convex polygonal domain. The coefficient matrix $a = (a_{ij})$ is symmetric and uniformly positive definite, $a_{ij} \in L^\infty(\Omega)$, $b \in L^\infty(\Omega)$ and $f$ is a given function in $L^2(\Omega)$.