INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 2, Pages 240–254

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Computing and Information

A NEW COLLOCATION METHOD FOR SOLVING CERTAIN HADAMARD FINITE-PART INTEGRAL EQUATION

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Abstract. In this paper, we study a new nodal-type trapezoidal rule for approximating Hadamard finite-part integrals, and its application to numerical solution of certain finite-part integral equation. We start with a nodal-type trapezoidal rule discussed in [21], and then establish its error expansion analysis, from which a new nodal-type trapezoidal rule with higher order accuracy is proposed and corresponding error analysis is also obtained. Based on the proposed rule, a new collocation scheme is then constructed to solve certain finite-part integral equation, with the optimal error estimate being rigorously derived. Some numerical experiments are also performed to verify the theoretical results.

Key words. Hadamard finite-part integral equation, quadrature rule, collocation method, error analysis.

1. Introduction

We consider the following finite-part integral

$$\mathcal{L}u(x) := \;=\!\!\!\!\!\!\int_a^b \frac{u(y)}{|y-x|^{1+2s}}\,dy, \quad x \in (a,b), \tag{1}$$

where $s \in (0,1)$ is the singularity index. The integral (1) is divergent in the classic Riemann sense, and should be understood in the Hadamard finite-part sense. There are several equivalent definitions for this finite-part integral in the literatures [15], and we here adopt the following definition:

$$\mathcal{L}u(x) = \lim_{\epsilon\to 0}\left(\int_{\Omega_\epsilon(x)} \frac{u(y)}{|y-x|^{1+2s}}\,dy - \frac{\epsilon^{-2s}u(x)}{s}\right), \quad x \in (a,b), \tag{2}$$

where x is the singular point and $\Omega_\epsilon(x) = (a,b)\backslash(x-\epsilon, x+\epsilon)$. A function $u(y)$ is said to be finite-part integrable with respect to the weight $|y-x|^{-1-2s}$ if the limit on the right-hand side of (2) exists. Assuming u is absolutely integrable on (a,b),then a sufficient condition for $u(x)$ to be finite-part integrable is that $u(x)$ is α-Hölder continuous for some $\alpha \in (2s,1)$ on (a,b) if $s \in (0,1/2)$, and $u'(x)$ is α-Hölder continuous for some $\alpha \in (2s-1,1)$ on (a,b) if $s \in [1/2,1)$.

Integrals of this kind appear in many practical problems related to aerodynamics, wave propagation or fluid mechanics, mostly with relation to boundary element methods and finite-part integral equations. Numerous work has been devoted in developing the efficient numerical evaluation method, such as Gaussian (GS) rule [6, 7], Newton-Cotes (NC) rule [9, 12, 14, 17, 20, 21], and some other rules [2, 3, 5]. Amongst them, NC rule is a popular one due to its ease of implementation and flexibility of mesh. NC rule is constructed by replacing u by its Lagrange interpolation in (1), and can be classified into two types: grid-type and nodal-type. The way of distinguishing one type from another is the choice of the singular point's location. Grid-type takes the singular point being located in the interior of a certain grid and nodal-type forces the singular point to be a certain nodal one.

Received by the editors XXX.
2000 *Mathematics Subject Classification.* 65R20, 65N30.

There are some other differences between these two type of rules. Amongst those, a major one is that the two rules are based on different definitions of the finite-part integrals (1) respectively. Since Lagrange interpolation is smooth in the interior of every grid, we can use the definition (2) directly to design grid-type NC rules. However, Lagrange interpolation is only continuous at the nodal points and the definition (2) is invalid to produce nodal-type NC rules, especially for $s \geq 1/2$, due to the rigorous regularity requirement on u for the definition (2). To overcome such problem, one often should adopt the following definitions [15]:

$$
\begin{aligned}
\mathcal{L}^- u(x) &= \lim_{\epsilon\to 0}\left(\int_a^{x-\epsilon} \frac{u(y)}{(x-y)^{1+2s}}\,dy + r^-(x)\right),\\
\mathcal{L}^+ u(x) &= \lim_{\epsilon\to 0}\left(\int_{x+\epsilon}^{b} \frac{u(y)}{(y-x)^{1+2s}}\,dy + r^+(x)\right),
\end{aligned} \tag{3}
$$

where

$$
r^-(x) = \begin{cases} \frac{\epsilon^{-2s}}{-2s} u(x^-), & s < 1/2,\\ -\epsilon^{-1}u(x^-) - \ln\epsilon\, u'(x^-), & s = 1/2,\\ \frac{\epsilon^{-2s}}{-2s} u(x^-) - \frac{\epsilon^{1-2s}}{1-2s} u'(x^-), & s > 1/2, \end{cases}
$$

$$
r^+(x) = \begin{cases} \frac{\epsilon^{-2s}}{-2s} u(x^+), & s < 1/2,\\ -\epsilon^{-1}u(x^+) + \ln\epsilon\, u'(x^+), & s = 1/2,\\ \frac{\epsilon^{-2s}}{-2s} u(x^+) + \frac{\epsilon^{1-2s}}{1-2s} u'(x^+), & s > 1/2, \end{cases}
$$

and $u(x^-)$ and $u(x^+)$ denote the left and right limits of u at x respectively. Obviously, if u is smooth enough, then $\mathcal{L}u(x) = \mathcal{L}^-u(x) + \mathcal{L}^+u(x)$.

It's well-known that the accuracy of NC rule with kth order piecewise polynomial interpolant for the usual Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. However, the rule is less accurate for finite-part integral (1) due to the hyper-singularity of the kernel. For example, general error analysis shows that the accuracy of both types of rules are $O(h^{k+1-2s})$ [4, 8, 9, 12, 14, 18]. A way of obtaining higher order accuracy for grid-type rule is to study its superconvergence property. This property implies that one can get higher order accuracy on the condition that the singular point coincides with some *a priori* known point. A series of outstanding works have been devoted to this field [11, 13, 16, 17, 18, 22, 23].

One goal of this paper is to study a higher order nodal-type rule for evaluation of (1). We start with a nodal-type trapezoidal rule ($k = 1$) investigated in [21]. Instead of estimating the error directly, we turn to analyze its error expansion. Once this expansion is established, a new nodal-type rule can be proposed by making a slight modification on the original one. As discussed in [21], the accuracy of the original rule is always $O(h^{2-2s})$. Excitingly, the new rule behaves more accurate, it reaches $O(h^{4-2s})$ if the singular point is far away from the endpoints, and $O(h^{3-2s})$ if very close to the endpoints, which is at least one order higher than the original rule.

A motivation to study the nodal-type NC rule is to solve the corresponding Hadamard finite-part integral equation defined by

$$
\begin{cases} \mathcal{L}u(x) = f(x), & x \in (a,b),\\ u(a) = u_a, u(b) = u_b. \end{cases} \tag{4}
$$