INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 2, Pages 255–275

AN EMBEDDED SDG METHOD FOR THE CONVECTION-DIFFUSION EQUATION

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Abstract. In this paper, we present an embedded staggered discontinuous Galerkin method for the convection-diffusion equation. The new method combines the advantages of staggered discontinuous Galerkin (SDG) and embedded discontinuous Galerkin (EDG) method, and results in many good properties, namely local and global conservations, free of carefully designed stabilization terms or flux conditions and high computational efficiency. In applying the new method to convection-dominated problems, the method provides optimal convergence in potential and suboptimal convergence in flux, which is comparable to other existing DG methods, and achieves L^2 stability by making use of a skew-symmetric discretization of the convection term, irrespective of diffusivity. We will present numerical results to show the performance of the method.

Key words. Embedded method, staggered discontinuous Galerkin method, convection-diffusion equation.

1. Introduction

Discontinuous Galerkin (DG) methods were first introduced by Reed and Hill for solving hyperbolic equations [40]. The DG methods have been proven superior to the classical continuous Galerkin (CG) methods for hyperbolic problems. In the past two decades, DG methods have also been applied to second-order elliptic problems. A comprehensive study on DG methods for elliptic problems is given in [1]. The original DG methods for elliptic problems, using polynomial approximations of degree k for both the potential and the flux, converge with optimal order $k + 1$ for the potential but suboptimal order k for the flux. While the same orders of convergence can be obtained by using classical CG finite element methods, the DG methods give rise to a discrete problem with a higher number of degrees of freedom. DG methods have therefore been criticized for its high computation cost and judged to be not being particularly useful for elliptic problems. Later, the hybridizible discontinuous Galerkin (HDG) method was introduced for solving elliptic problems [23]. The HDG method provides optimal orders of convergence for both the potential and the flux in L^2 norm. Moreover, superconvergence can be obtained for the potential through a local postprocessing technique.

In recent years, there are active developments of DG methods for problems in fluid dynamics and wave propagations, see for example [6, 22, 24, 27, 28, 32, 36, 41, 42, 37]. On the other hand, staggered meshes bring the advantages of reducing numerical dissipation in computational fluid dynamics [2, 3, 31], and numerical dispersion in computational wave propagation [9, 10, 11, 12, 13, 14, 17]. Combining the ideas of DG methods and staggered meshes, a new class of staggered discontinuous Galerkin (SDG) methods was proposed for approximations of Stokes system [34], convection-diffusion equation [19], and incompressible Navier-Stokes equations [7]. The new class of SDG methods possesses many good properties, including local and global conservations, stability in energy, and optimal convergence. For a more

Received by the editors XXX.
2000 *Mathematics Subject Classification.* 65N12, 65N30.

complete discussion on the SDG method, see also [11, 12, 13, 14, 18, 19, 35] and the references therein.

In [15, 16], it was shown that the SDG method can be regarded as a limit of the HDG method. The SDG method can be obtained from the HDG method by setting the stabilization parameter on a set of edges to be zero and letting the parameter on another set of edges to infinity. As a result, the SDG method inherits the advantages of the HDG method, including superconvergence through the use of a local postprocessing technique. Furthermore, in the SDG method for incompressible Navier-Stokes equations [7], using the postprocessing and a spectro-consistent discretizations with a novel splitting of the diffusion and the convection term, stability in L^2 energy is achieved.

The embedded discontinuous Galerkin (EDG) method was first introduced for solving the linear shell problems [30]. Later, an EDG method for solving second order elliptic problems was discussed and analyzed in [26]. The EDG method was obtained from HDG method by enforcing strong continuity for hybrid unknowns [23]. This greatly reduces the number of degrees of freedom in the globally coupled system and makes the EDG method has a higher computational efficiency compared with other DG methods. As a tradeoff for this advantage, the EDG method is not locally conservative and loses the optimal convergence in the flux achieved by the HDG method [38]. The loss in accuracy makes the EDG method a less attractive candidate compared with the HDG method. However, the optimal order of convergence for HDG method is also lost in the case of convection-dominated problems as shown the numerical examples [25]. In this case, the EDG method becomes appealing alternative to all other DG methods including the HDG method, since it has a higher computational efficiency and the same orders of convergence. On the other hand, compared with the CG finite element method, the EDG method provides the same sparsity structure of the stiffness matrix after static condensation, whilist the EDG method is more robust, accurate and stable than the CG finite element method in convection-dominated problems. We remark that the multiscale discontinuous Galerkin (MDG) method [5, 33] are related to the EDG method, which is originally proposed for the convection-diffusion problems. The MDG method and the EDG method are both designed for a globally continuous approximation of the solution and can give rise to identical schemes. Recently, the EDG method has been proposed on Euler equations and Navier-Stokes equations [38, 39]. Due to the advantages shared with other DG methods and the high computational efficiency compared with other DG methods, the EDG method has also been applied to challenging problems in computational fluid dynamics, such as implicit large eddy simulation [29].

In this paper, we propose a combination of the SDG method and the EDG method for the convection-diffusion equation. The new method seeks approximations in the SDG locally conforming finite element spaces, which gives rise to a flux formulation without introducing any carefully designed stabilization terms or flux conditions as in other DG methods. The new method further reduces the size of the global discrete problem compared with SDG method by restricting the numerical approximation for the primal unknown to lie in a proper subspace of the SDG finite element space. Moreover, the new method inherits the stability in L^2 energy thanks to spectro-consistent discretizations in the SDG method. The convergence is optimal with order $k+1$ for the unknown function and suboptimal with order k for its gradient, which are comparable to all other DG methods for convection-dominated problems.