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A TIME SECOND-ORDER CHARACTERISTIC FINITE ELEMENT METHOD FOR NONLINEAR ADVECTION-DIFFUSION EQUATIONS

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Abstract. In this paper, a new time second-order characteristic finite element method is proposed and analyzed for solving the advection-diffusion equations involving a nonlinear right side term. In order to obtain a time second-order characteristic scheme, the global derivative term transferred from the time derivative and advection terms is discretized by using difference operator along the characteristics, the diffusion term is discretized by the average operator along the characteristics, while specially a second-order extrapolation along the characteristics is applied to the right side nonlinear function. We analyze and prove that the proposed scheme for the nonlinear advection-diffusion equations has second order accuracy in time step size, which improves the first order accuracy in time of the classical characteristic methods. The proposed characteristic FEM scheme allows to use large time step sizes in computation. Numerical tests are taken to show the accuracy of our proposed scheme, and the case of single-species population dynamics is further simulated and analyzed by using our method and numerical results show its advantage and effectiveness.

Key words. Nonlinear advection-diffusion equations, characteristic method, characteristic average, characteristic extrapolation, time second-order, error estimate, population dynamics.

1. Introduction

Nonlinear advection-diffusion equations have been widely applied in science and engineering computing such as in heat and mass transfer, oil reservoir simulation, groundwater modelling, atmospheric pollution, aerodynamics, biological population, and so on (see, for examples, [3, 4, 6, 9, 11, 20, 21, 22, 23, 24], etc). The nonlinear right side terms are of significance, which describe the different important physical and chemical processes such as the nonlinear reaction in physics, the biological nonlinear predator-prey and competition interaction in biological population, and the nonlinear coagulation process in atmospheric aerosol dynamics and so on. The nonlinear function of population dynamics results from the interaction of birth rate, death rate and the environment factor, which can lead to the density change of population in the global dynamics. Studying such problems has been playing more and more important role in discovering the relation between species and their environment, and in understanding the dynamic processes involved in such areas as the spread and control of diseases and viruses, predator-prey and competition interaction, evolution of pesticide-resistant strains, biological pest control, and so on. The study can also be used to describe, predict and adjust the developing trend of the population. So this research subject has an actual meaning very much.

Classical finite element methods and finite difference methods can resolve the diffusion problems well when diffusion dominates the physical process. However, solving advection-dominated diffusion equations presents non-physical oscillations or excessive numerical dispersions at steep fronts. In the framework of numerical

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solutions of advection-dominated diffusion problems, the discretization along characteristics is a good technique, which takes advantage of the physics characteristics of the convection-diffusion equations, it can not only reduce the non-physical oscillation and excessive numerical dispersion but also has no stability constraints required on the time step. Douglas and Russell [10] first proposed and analyzed the modified method of characteristics (MMOC) methods for solving one dimensional convection diffusion problems, where they combined a time first order characteristics scheme with both finite differences and classical Lagrange finite elements for time dependent convection-diffusion problems. Lately, combined with the mixed finite element methods, [2, 12, 16, 25] developed the characteristic mixed element methods for convection-diffusion problems, miscible displacement flows and immiscible displacement flows. Further developments of characteristics methods were carried out in [1, 5, 7, 17, 19], etc. The characteristic methods avoid the grid distortion, significantly reduce the truncation errors in time and eliminate the excessive numerical dispersions. However, these above characteristic methods are only of time first order accuracy. In order to improve the accuracy, recently, for nonlinear atmospheric aerosol dynamics, [18] developed a time second-order characteristic finite element scheme, where the mathematical model is the advection equation with a nonlinear coagulation integration right side term, but the equation has no diffusion term. Hence, there is of importance to develop and analyze the time second-order characteristic finite element methods to solve nonlinear advection diffusion problems with nonlinear right side terms.

In this paper, we develop and analyze a time second-order characteristic finite element method for solving nonlinear advection-diffusion problems, the time derivative term and the advection term are first transferred into the global derivative term and it is then discretized by the central difference operator along the characteristic curve. The diffusion term is approximated by the second-order average operator along the characteristic curve. For treating the nonlinear right side term, the second-order extrapolation along the characteristics is applied, where two previous level values are used along the characteristics. Both the average operator and the extrapolation are different from the normal Crank-Nicoson average and extrapolation along the time direction, they are proposed to be along the characteristics in our scheme. The developed scheme has second-order accuracy in time and can provide efficiently high accuracy solutions when using large time step sizes. Using the theory of variation and prior estimates, we analyze theoretically the developed characteristic finite element method for the nonlinear advection diffusion equations. We prove our developed characteristic scheme to have the error estimate of $O((\Delta t)^2 + h^l)$. Numerical tests for nonlinear advection diffusion equations first show that our methods have the second-order accuracy in time, which confirm the theoretical analysis results. Meanwhile, we compute the moving of a sharp front gradient and our scheme performs excellently. Finally, in simulation of the population dynamics, considering the one-dimensional single-species spatio-temporal population dynamic models which demonstrate the intricate interplay between mixing, advection, boundary conditions and other factors which are critical for the persistence of a population. We assume the population model is the nonlinear logistic growth. The change of density occurs as a result of the nonlinear logistic death-birth processes and of the spatial movement of organisms due to diffusion and advection. With different values of environmental and biological parameters, the simulated results show that changes of parameters can make the different shapes and propagations of population density distribution, thus we