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NUMERICAL ANALYSIS OF AN ENERGY-CONSERVATION SCHEME FOR TWO-DIMENSIONAL HAMILTONIAN WAVE EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper, an energy-conservation scheme is derived and analysed for solving Hamiltonian wave equations subject to Neumann boundary conditions in two dimensions. The energy-conservation scheme is based on the blend of spatial discretisation by a fourth-order finite difference method and time integration by the Average Vector Field (AVF) approach. The spatial discretisation via the fourth-order finite difference leads to a particular Hamiltonian system of second-order ordinary differential equations. The conservative law of the discrete energy is established, and the stability and convergence of the semi-discrete scheme are analysed. For the time discretisation, the corresponding AVF formula is derived and applied to the particular Hamlitonian ODEs to yield an efficient energy-conservation scheme. The numerical simulation is implemented for various cases including a linear wave equation and two nonlinear sine-Gordon equations. The numerical results demonstrate the spatial accuracy and the remarkable energy-conservation behaviour of the proposed energy-conservation scheme in this paper.

Key words. Two-dimensional Hamiltonian wave equation, finite difference method, Neumann boundary conditions, energy-conservation algorithm, average vector field formula.

1. Introduction

The theme of this paper is the numerical analysis of an energy-conservation scheme for the following Hamiltonian wave equation in two-dimensional space:

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy}) = f(u), \quad (x, y, t) \in \mathbf{\Omega} \times (t_0, T], \\ u(x, y, t_0) = \varphi(x, y), \ u_t(x, y, t_0) = \phi(x, y), \quad (x, y) \in \bar{\mathbf{\Omega}}, \end{cases} \tag{1}$$

where the function $f(u)$ is the negative derivative of a potential energy $V(u)$, and $\mathbf{\Omega} = (x_l, x_r) \times (y_d, y_u)$. The initial functions φ and ϕ are wave modes or kinks and their velocity, respectively (see, e.g. [22, 23, 38]). Here, we suppose that the system (1) is supplemented with the homogenous *Neumann boundary conditions:*

$$\frac{\partial u}{\partial x}\Big|_{x=x_r,x_l} = 0, \quad y_d \le y \le y_u, \quad \frac{\partial u}{\partial y}\Big|_{y=y_d,y_u} = 0, \quad x_l \le x \le x_r, \quad \forall t \in [t_0, T]. \tag{2}$$

It is noted that the conservation of the energy is an essential property of the Hamiltonian system (1)-(2), which can be expressed as follows:

$$E(t) \equiv \frac{1}{2} \int_{\mathbf{\Omega}} \Big[u_t^2 + a^2 \big(u_x^2 + u_y^2 \big) + 2V(u) \Big] \mathrm{d}x\mathrm{d}y = E(t_0). \tag{3}$$

Therefore, it is very important to design numerical schemes that can precisely preserve a discrete energy. This class of schemes is called conservative schemes. They often yield physically correct results and numerical stability [8].

In the literature, a considerable number of numerical schemes has been presented for solving the two-dimensional Hamiltonian wave equations (1). For example,

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Bratsos [5] applied the method of lines to solve system (1). In [13], the authors constructed a high-order compact alternating direction implicit scheme. Based on the blend of spatial discretisation by different finite difference methods and time integration by predictor-corrector schemes, the authors (see, e.g. [9, 12, 17]) investigated nonlinear Hamiltonian wave equations numerically. Dehghan et al. (see, e.g. [14, 15, 34, 35]) proposed the meshless local Petrov-Galerkin (MLPG) method, the meshless local radial point interpolation (LRPI) method, the dual reciprocity boundary element (DRBE) method and the meshless local boundary integral equation (LBIE) method for solving the two-dimensional sine-Gordon equations. In [29, 30, 31], Liu et al. first formulated the system (1) as an abstract second-order ODEs in a suitable infinite-dimensional function space and then proposed and analysed a class of arbitrarily high-order time-stepping methods for solving Hamiltonian system (1). Mei et al. [33] presented an extension of the finite-energy condition for Runge–Kutta–Nyström-type integrators solving nonlinear wave equations. The other numerical approaches also were investigated, such as the finite element method (see, e.g. [1, 2]), the perturbation method [24] and the spectral method (see, e.g. [3, 28]). On the other hand, splitting methods (see, e.g. [9, 11, 16, 17, 37]) have been developed to investigate the numerical solutions of the evolutional PDEs in multi-dimensional case. Sheng et al. [38] provided highly efficient splitting cosine schemes for the two-dimensional sine-Gordon equations by using a linearly implicit sequential splitting or Strangs splitting. In a recent study of Hamiltonian wave equations, using the finite difference method and the average vector field (AVF) method, the authors consider a class of energy-conservation methods for one-dimensional Hamiltonian wave equations (see [26]). However, there are very few studies to pay attention to designing and analysing high-order energy-conserving schemes for the two-dimensional Hamiltonian wave equations subject to Neumann boundary conditions. Hence, in this work, we devote to constructing and analysing a novel and efficient energy-conservation scheme for the Hamiltonian wave equations with Neumann boundary conditions (1)-(2) in two dimensions. According to the homogenous Neumann boundary conditions and the structure of equation (1), we derive and analysis a kind of fourth-order finite difference operators to discrete the spatial derivatives of the system (1)-(2). In such a way, the conserved PDEs can be converted into a particular Hamiltonian system of ODEs expressed in the form $Aw''(t) + Bw(t) = \tilde{f}\big(w(t)\big)$, and the Hamiltonian of the obtained ODEs can be thought of as the approximate energy of the original continuous system. This motivates us to consider the AVF approach to the discretisation of the derived Hamiltonian ODEs' system in time. Therefore, incorporating the fourth-order finite difference discretisation in space with the AVF time integrator yields a novel and efficient energy-conservation numerical scheme for the two-dimensional Hamiltonian wave equations (1)-(2).

The paper is organised as follows. In Section 2, a particluar Hamiltonian ODEs' system is obtained by applying the fourth-order finite difference operators to discrete the spatial derivatives of the nonlinear system (1)-(2). Then the conservation law, the stability and the convergence of the semi-discrete scheme are rigorously analysed. Section 3 is devoted to describing in detail the idea of the AVF formula for the derived Hamiltonian system of ODEs. In Section 4, numerical experiments on a linear Klein-Gordon equation and two sine-Gordon equations are implemented and the numerical results show the convergence order of the spatial discretisation and the excellent conservative property of the proposed numerical scheme. Section 5 is concerned with conclusions.