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FEM-ANALYSIS ON GRADED MESHES FOR TURNING POINT PROBLEMS EXHIBITING AN INTERIOR LAYER

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Abstract. We consider singularly perturbed boundary value problems with a simple interior turning point whose solutions exhibit an interior layer. These problems are discretised using higher order finite elements on layer-adapted graded meshes proposed by Liseikin. We prove ε-uniform error estimates in the energy norm. Furthermore, for linear elements we are able to prove optimal order ε-uniform convergence in the L^2-norm on these graded meshes.

Key words. Singular perturbation, turning point, interior layer, layer-adapted meshes, higher order finite elements.

1. Introduction

We consider singularly perturbed boundary value problems of the type

$$\begin{aligned} -\varepsilon u''(x) + a(x)u'(x) + c(x)u(x) &= f(x) \qquad \text{in } (-1,1), \\ u(-1) = \nu_{-1}, \quad u(1) &= \nu_1, \end{aligned} \tag{1a}$$

where $0 < \varepsilon \ll 1$ is a small parameter and a, c, f are sufficiently smooth with

$$a(x) = -(x - x_0)b(x), \qquad b(x) > 0, \qquad c(x) \geq 0, \qquad c(x_0) > 0 \tag{1b}$$

for a point $x_0 \in (-1, 1)$. Thus, the solution of (1) exhibits an interior layer of "cusp"-type at the simple interior turning point x_0.

In the literature (see e.g. [2], [4, p. 95], [7, Lemma 2.3]) the bounds for such interior layers are well known. We have

$$\left| u^{(i)}(x,\varepsilon) \right| \leq C \left(1 + \left(\varepsilon^{1/2} + |x - x_0| \right)^{\lambda - i} \right) \tag{2}$$

where the parameter λ satisfies $0 < \lambda < \bar{\lambda} := c(x_0)/|a'(x_0)|$. The estimate also holds for $\lambda = \bar{\lambda}$, if $\bar{\lambda}$ is not an integer. Otherwise there is an additional logarithmic factor, see references cited above. For convenience we assume $x_0 = 0$ in the following.

In the last decades a multitude of numerical methods has been developed to solve singularly perturbed problems with turning points and interior layers. For a general review we refer to [6]. Many authors have considered finite difference methods. A selection of possible schemes for problems of the form (1) may be found in [3] and the references therein. Also some layer-adapted meshes have been proposed to handle interior layers of "cusp"-type. As an example Liseikin [4] proved the ε-uniform first order convergence of an upwind finite difference method on special graded meshes. Moreover, Sun and Stynes [7] studied finite elements on a piecewise uniform mesh.

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We shall also analyse the finite element method, but on the graded meshes proposed by Liseikin which are described by the mesh generating function

$$\varphi(\xi,\varepsilon)=\begin{cases}\left(\varepsilon^{\alpha/2}+\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha}-\varepsilon^{1/2} & \text{for } 0\le\xi\le 1,\\ \varepsilon^{1/2}-\left(\varepsilon^{\alpha/2}-\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha} & \text{for } 0\ge\xi\ge -1,\end{cases}$$

where $0<\alpha\le\lambda$. In order to handle these meshes, we adapt some basic ideas from [4, pp. 243–244]. While the strategy of Sun and Stynes in [7, Section 5] is restricted to linear finite elements, our approach is more general. Thus, we are able to treat finite elements of higher order as well. Also note that recently, in [1], a similar approach was used to study the streamline diffusion finite element method on the piecewise uniform mesh of Sun and Stynes.

Under certain assumptions, we prove ε-uniform convergence in the energy norm of the form

$$|||u-u_N|||_\varepsilon\le CN^{-k}$$

for finite elements of order k, where C may depend on α and k, see Theorem 3.5. On the basis of a supercloseness result we also give an optimal error estimate in the L^2-norm of the form

$$\|u-u_N\|\le CN^{-2}$$

for linear finite elements, see Theorem 3.10. Numerical experiments confirm our theoretical results.

Notation: In this paper C denotes a generic constant independent of ε and the number of mesh points. Furthermore, for an interval I the usual Sobolev spaces $H^1(I)$, $H_0^1(I)$, and $L^2(I)$ are used. The spaces of continuous and k times continuously differentiable functions on I are written as $C(I)$ and $C^k(I)$, respectively. Let $(\cdot,\cdot)_I$ denote the usual $L^2(I)$ inner product and $\|\cdot\|_I$ the $L^2(I)$-norm. We will also use the supremum norm on I given by $\|\cdot\|_{\infty,I}$ and the seminorm in $H^1(I)$ given by $|\cdot|_{1,I}$. If $I=(-1,1)$, the index I in inner products, norms, and seminorms will be omitted. Additionally, for all $v\in H^1((-1,1))$ we define a weighted energy norm by

$$|||v|||_\varepsilon:=\left(\varepsilon\,|v|_1^2+\|v\|^2\right)^{1/2}.$$

Further notation will be introduced later at the beginning of the sections where it is needed.

2. The graded meshes proposed by Liseikin

The basic idea of Liseikin is to find a transformation $\varphi(\xi,\varepsilon)$ that eliminates the singularities of the solution when it is studied with respect to ξ. In our case the approach can be condensed to the task to find $\varphi:[0,1]\to[0,1]$ such that

$$\varphi'\left(\varphi+\varepsilon^{1/2}\right)^{\lambda-1}\le C,\qquad \varphi(0)=0,\qquad \varphi(1)=1. \tag{3}$$

The outcome of this approach is the mesh generating function

$$\varphi(\xi,\varepsilon)=\begin{cases}\left(\varepsilon^{\alpha/2}+\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha}-\varepsilon^{1/2} & \text{for } 0\le\xi\le 1,\\ \varepsilon^{1/2}-\left(\varepsilon^{\alpha/2}-\xi\left[(1+\varepsilon^{1/2})^{\alpha}-\varepsilon^{\alpha/2}\right]\right)^{1/\alpha} & \text{for } 0\ge\xi\ge -1,\end{cases} \tag{4}$$

where $0<\alpha\le\lambda$. By construction we have $\varphi(0,\varepsilon)=0$ and $\varphi(\pm1,\varepsilon)=\pm1$. Note that Liseikin derived the same transformation indirectly. Based on the principle of