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IMPROVED ERROR ESTIMATION FOR THE PARTIALLY PENALIZED IMMERSED FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS

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Abstract. This paper is for proving that the partially penalized immersed finite element (PPIFE) methods developed in [25] converge optimally under the standard piecewise H^2 regularity assumption for the exact solution. In energy norms, the error estimates given in this paper are better than those in [25] where a stronger piecewise H^3 regularity was assumed. Furthermore, with the standard piecewise H^2 regularity assumption, this paper proves that these PPIFE methods also converge optimally in the L^2 norm which could not be proved in [25] because of the excessive H^3 regularity requirement.

Key words. Interface problems, immersed finite element methods, optimal convergence, discontinuous coefficients, finite element spaces, interface independent mesh, regularity.

1. Introduction

In this article, we establish better error estimates for the numerical solutions generated by the partially penalized immersed finite element (PPIFE) methods [25] for the interface problem governed by the second-order elliptic equation:

$$-\nabla \cdot (\beta \nabla u) = f, \quad \text{in } \Omega^- \cup \Omega^+, \tag{1a}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1b}$$

where, without loss of generality, the domain $\Omega \subseteq \mathbb{R}^2$ is divided by an interface curve Γ into two subdomains Ω^- and Ω^+, and the coefficient β is a piecewise positive constant function such that

$$\beta(X) = \begin{cases} \beta^- & \text{for } X \in \Omega^-, \\ \beta^+ & \text{for } X \in \Omega^+. \end{cases}$$

In addition, the exact solution u satisfies the following jump conditions across the interface

$$[u]_\Gamma \quad := \quad u^-|_\Gamma - u^+|_\Gamma = 0, \tag{2}$$

$$\left[\beta \nabla u \cdot \mathbf{n}\right]_\Gamma \quad := \quad \beta^- \nabla u^- \cdot \mathbf{n}|_\Gamma - \beta^+ \nabla u^+ \cdot \mathbf{n}|_\Gamma = 0, \tag{3}$$

where $\mathbf{n}$ is the unit normal vector to the interface Γ. For the sake of simplicity, as in [25], we assume the interface Γ is a C^2-curve and does not intersect $\partial\Omega$.

The immersed finite element (IFE) method is developed to solve the interface problem (1)-(3) on an interface independent mesh, if desirable, a simple structured (Cartesian) mesh can be used. The key idea of this method is to utilize Hsieh-Clough-Tocher type macro polynomials [3, 6], i.e., the piecewise polynomials constructed according to the jump conditions on interface elements to capture the jump behaviors of the exact solutions [2, 9, 16, 22], while standard polynomials are used on all the non-interface elements. The global IFE functions such as those used in [16, 22, 25] are, in general, not continuous across the interface edges, even though the continuity at the mesh nodes is imposed. The partially penalized IFE (PPIFE)

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methods developed in [25] employed the interior penalties [8] on interface edges to control the adverse effects from those discontinuities so that these PPIFE methods converge optimally in a certain energy norm. Penalties are also used in Cut-FEMs [4, 14] mainly for enhancing jump conditions across the interface. IFE methods for interface problems associated with other types of PDEs or jump conditions as well as the applications can be found in [1, 5, 10, 11, 17, 18, 19, 20, 21, 23, 24, 26, 28], to name just a few.

The authors in [25] employed a piecewise H^3 regularity assumption for the exact solution to the interface problem to prove the optimal convergence of the PPIFE solutions. However, given the body force term $f \in L^2(\Omega)$, the exact solution to (1)-(3) only has the piecewise H^2 regularity [7] in general. This motivates us to investigate whether the PPIFE methods developed in [25] can converge optimally under the standard piecewise H^2 regularity assumption instead of the excessive piecewise H^3 regularity. Towards this goal, we introduce a new energy norm that is stronger than the one used in [25]. Inspired by [13], we derive an estimate for the IFE interpolation error gauged by this energy norm on a patch of each interface element. Furthermore, the bilinear form in the PPIFE method has both the continuity and coercivity in this energy norm. These properties enable us to derive an error bound for the PPIFE solution in the energy norm under the standard piecewise H^2 regularity assumption. As an important consequence, the improved estimation further enables us to show the optimal convergence in the L^2 norm, which, to our best knowledge, has not been established in the literature for the PPIFE methods.

This article consists of four additional sections. The next section reviews some notations from [25] which will be also used in this article. In Section 3, we introduce the patches for the interface elements and recover the approximation capabilities of IFE spaces on these patches. In Section 4 we show the optimal convergence of the PPIFE solutions. Finally, we make some conclusions in Section 5.

2. Notations and IFE Spaces

We herein adopt some notations from [25]. For every measurable open set $\tilde{\Omega} \subseteq \Omega$, we let $\tilde{\Omega}^s := \tilde{\Omega} \cap \Omega^s, s = \pm$, and we let $W^{k,p}(\tilde{\Omega})$ be the standard Sobolev space on $\tilde{\Omega}$ with the standard norm $\|\cdot\|_{k,p,\tilde{\Omega}}$ and the semi-norm $|v|_{k,p,\tilde{\Omega}}$. When $\tilde{\Omega}^s \neq \emptyset$, $s = \pm$, we let the related Sobolev norms and semi-norms be

$$\|\cdot\|^2_{k,p,\tilde{\Omega}} = \|\cdot\|^2_{k,p,\tilde{\Omega}^-} + \|\cdot\|^2_{k,p,\tilde{\Omega}^+}, \qquad |\cdot|^2_{k,p,\tilde{\Omega}} = |\cdot|^2_{k,p,\tilde{\Omega}^-} + |\cdot|^2_{k,p,\tilde{\Omega}^+}.$$

Furthermore, we introduce the following spaces on $\tilde{\Omega}$ in the case $\tilde{\Omega}^s \neq \emptyset$, $s = \pm$:

$$PW^{k,p}(\tilde{\Omega}) = \{u : u|_{\tilde{\Omega}^s} \in W^{k,p}(\tilde{\Omega}^s),\ s = \pm;\ [u] = 0,\ [\beta \nabla u \cdot \mathbf{n}_\Gamma] = 0 \text{ on } \Gamma \cap \tilde{\Omega}\},$$

for suitable k and p such that involved qualities on $\Gamma \cap \tilde{\Omega}$ are well defined. As usual, we will drop p from the pertinent norms and semi-norms for $H^k(\tilde{\Omega}) = W^{k,2}(\tilde{\Omega})$ and $PH^k(\tilde{\Omega}) = PW^{k,2}(\tilde{\Omega})$.

We let $\mathcal{T}_h$ be a triangular or a rectangular mesh for the domain $\Omega \subset \mathbb{R}^2$ and let $\mathcal{N}_h$ be the collection of the nodes in the mesh $\mathcal{T}_h$. We denote the sets of interface elements and non-interface elements by $\mathcal{T}_h^i$ and $\mathcal{T}_h^n$. Also, we denote the set of interior edges by $\mathring{\mathcal{E}}_h$, the interior interface edges by $\mathring{\mathcal{E}}_h^i$ and the interior non-interface edges by $\mathring{\mathcal{E}}_h^n$, respectively. For each element $T \in \mathcal{T}_h$, we define its index set as $\mathcal{I}_T = \{1, 2, 3\}$ when T is triangular, but $\mathcal{I}_T = \{1, 2, 3, 4\}$ when T is rectangular. Given each T, let A_i, $i \in \mathcal{I}_T$ be the vertices of T, and the interface partitions the