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FORMULAS OF NUMERICAL DIFFERENTIATION ON A UNIFORM MESH FOR FUNCTIONS WITH THE EXPONENTIAL BOUNDARY LAYER

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Abstract. It is known that the solution of a singularly perturbed problem corresponds to the function with large gradients in a boundary layer. The application of Lagrange polynomial on a uniform mesh to interpolate such functions leads to large errors. To achieve the error estimates uniform with respect to a small parameter, we can use either a polynomial interpolation on a mesh which condenses in a boundary layer or we can use special interpolation formulas which are exact on a boundary layer component of the interpolating function. In this paper, we construct and study the formulas of numerical differentiation based on the interpolation formulas which are exact on a boundary layer component. We obtained the error estimates which are uniform with respect to a small parameter. Some numerical results validating the theoretical estimates are discussed.

Key words. Function of one variable, exponential boundary layer, formulas of numerical differentiation, an error estimate.

1. Introduction

Singularly perturbed problems are used for modeling convection-diffusion processes with dominant convection therefore the question of numerical solution of such problems is relevant. Solutions of singularly perturbed problems have large gradients in a boundary layer, thus application of classical difference schemes leads to large errors. To construct the difference schemes which converge uniformly with respect to a small parameter ε, there are two basic approaches based on papers A. M. Ilyin [1] and N. S. Bakhvalov [2].

In [1], it is proposed to construct difference schemes based on a fitting to the boundary layer component. These schemes are known as the exponential fitting schemes and were constructed in many works, for example in monographs [3,4] and the references therein. The schemes of arbitrarily high ε-uniform order of accuracy on a uniform mesh were constructed in [5,6]. According to [7] the exponential fitted scheme on a uniform mesh is ε-uniformly accurate in the case of an elliptic problem with regular boundary layers.

In [2], it is proposed to apply classical difference scheme on a special mesh that condenses in a boundary layer. This approach was developed in [8–15] and in a number of other works.

The question of construction of numerical differentiation formulas for functions with large gradients in a boundary layer is relevant too, because the use of classical formulas based on a differentiation of the Lagrange polynomial [16] leads to large errors. We can use the approaches used to construct difference schemes which converge ε-uniformly to create the acceptable numerical differentiation formulas.

The approach based on application of classical numerical differentiation formulas on meshes that condense in a boundary layer was investigated in [17–21] and some other works. In [17], an ordinary convection-diffusion equation is considered. The

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upwind scheme on the Shishkin and the Bakhvalov meshes with the property of ε-uniform convergence was used. To calculate the first derivative of the solution of differential problem, the authors used the solution of difference scheme and one-sided difference formula. The estimate of the relative error in a boundary layer and the estimate of the absolute error outside a boundary layer were obtained. These estimates are ε-uniform. In [20], this approach was applied to solve numerically a weakly coupled system of two singularly perturbed convection-diffusion second order ordinary differential equations on the Shishkin mesh. In [18, 19], the error of difference formulas on the Shishkin mesh for the derivatives of the solution of a singularly perturbed elliptic problem was investigated. In [21], the difference scheme on the Shishkin mesh for a linear singularly perturbed parabolic convection-diffusion problem was investigated. Similarly to [17] the error of the numerical differentiation formulas at mesh nodes was estimated.

In this paper, we study a problem of numerical differentiation on a uniform mesh by the use of a fitting of the difference formulas to the component responsible for the large gradients of the function in a boundary layer. The study of this approach is of interest for the following reasons. Difference schemes on uniform meshes are applicable to the numerical solution of a number of singularly perturbed problems as was mentioned above. Difference formulas for derivatives with the exponential fitting can be successfully applied to construct difference schemes and splines which converge ε-uniformly. It can be necessary in the case of initial or boundary conditions in a boundary layer region to approximate the first or the second derivatives. We applied such approach in [22, 23]. In [24], special difference formulas for approximation of derivatives were used to construct exact difference schemes but this method was not applied to singularly perturbed problems.

We assume that a function $u(x)$ has the decomposition:

$$u(x) = p(x) + \gamma\,\Phi(x), \qquad x \in [0,\,1], \tag{1}$$

where the functions $u(x)$, $p(x)$, $\Phi(x)$ are sufficiently smooth, the boundary layer component $\Phi(x)$ is known and responsible for the large gradients of the function $u(x)$, the function $p'(x)$ is uniformly bounded. We also assume that the constant γ and the function $p(x)$ are unknown but the estimates of certain derivatives of the function $p(x)$ are known.

In [25], the decomposition (1) of the solution of a singularly perturbed boundary value problem was investigated. The authors applied the decomposition to prove a uniform convergence of the difference scheme.

To construct the example of decomposition (1), we consider a singularly perturbed problem

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \qquad u(0) = A, \quad u(1) = B, \tag{2}$$

where $a(x) \geqslant \alpha_0 > 0$, $b(x) \geqslant 0$, $\varepsilon \in (0, 1]$, functions a, b, f are sufficiently smooth, the constant α_0 is separated from zero. It is known [25] that for small values of parameter ε a solution of the problem (2) has exponential boundary layer near $x = 0$ and the function $u(x)$ has the form (1). If we specify

$$\Phi(x) = e^{-a_0\,x/\varepsilon}, \quad a_0 = a(0), \quad \gamma = -\varepsilon u'(0)/a_0,$$

then there are the estimates $|p'(x)| \leqslant C_0$, $|\gamma| \leqslant C_0$, where the constant C_0 is independent of ε. In this case the derivatives $p^{(j)}(x)$, $j \geqslant 2$ can be unbounded for small value ε but the function $\gamma\Phi(x)$ is responsible for the main growth of $u(x)$ in a boundary layer. For this reason, we construct formulas of numerical differentiation that are exact on the function $\Phi(x)$. We also study an accuracy of such formulas