INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 4, Pages 626–646



# ERROR ANALYSIS OF AN IMMERSED FINITE ELEMENT METHOD FOR TIME-DEPENDENT BEAM INTERFACE PROBLEMS

MIN LIN

**Abstract.** This article presents an error analysis of a Hermite cubic immersed finite element (IFE) method for solving certain initial-boundary value problems (IBVP) modeling a time-dependent Euler-Bernoulli beam formed by multiple materials together with suitable jump conditions at material interfaces. The optimal convergence of this IFE method is shown by both theoretical proof and numerical simulations.

**Key words.** Interface problem, time-dependent beam model, IFE method, fully discrete, error analysis.

## 1. Introduction

In this paper, we present an error analysis of a Hermite cubic immersed finite element (IFE) method for solving interface problems related to a mathematical model for a time-dependent Euler-Bernoulli beam formed with multiple materials. Without loss of generality, we consider a beam of length 1 formed with two materials, and we assume its dynamics is modeled by the following initial-boundary value problem (IBVP) [23]:

$$\rho(x)u_{tt}(x,t) + \big(\beta(x)u_{xx}(x,t)\big)_{xx} = f(x,t),\ x \in (0,1)\backslash\{\alpha\},\ t \in (0,T], \tag{1a}$$

$$u(0,t) = b_1(t),\ u_x(0,t) = b_2(t),\ u(1,t) = b_3(t),\ u_x(1,t) = b_4(t), \tag{1b}$$

$$u(x,0) = g_1(x),\ u_t(x,0) = g_2(x), \tag{1c}$$

and the rigid connection condition across the material interface $\alpha$ as follows:

$$\begin{cases} \big[u(x,t)\big]_{x=\alpha} = 0, & \text{(continuity in the deflection)},\\ \big[\dfrac{\partial u(x,t)}{\partial x}\big]_{x=\alpha} = 0, & \text{(continuity in the bending angle)},\\ \big[\beta(x)\dfrac{\partial^2 u(x,t)}{\partial x^2}\big]_{x=\alpha} = 0, & \text{(continuity of the bending moment)},\\ \big[\dfrac{\partial(\beta(x)\frac{\partial^2 u(x,t)}{\partial x^2})}{\partial x}\big]_{x=\alpha} = 0, & \text{(continuity of the shear)}, \end{cases} \tag{1d}$$

where $u(x,t)$ is the transverse displacement of the beam at time $t$ and longitudinal coordinate $x$, $\rho(x)$ is the mass density, $\beta(x)$ is the bending modulus or stiffness parameter, and $f(x,t)$ is the distributed loading force. Note that $[w(x,t)]_{x=\alpha} := \lim_{x\to\alpha^+} w(x,t) - \lim_{x\to\alpha^-} w(x,t)$. For simplicity, we assume that the material parameters

Received by the editors on June 20, 2018 and, accepted on October 19, 2018.
2000 *Mathematics Subject Classification.* 65N15, 65N30, 65N50, 35R05.

$\rho(x)$ and $\beta(x)$ are both piecewise positive constant functions:

$$\rho(x) = \begin{cases} \rho^-, & x \in \Omega^-, \\ \rho^+, & x \in \Omega^+, \end{cases} \tag{1e}$$

$$\beta(x) = \begin{cases} \beta^-, & x \in \Omega^-, \\ \beta^+, & x \in \Omega^+, \end{cases} \tag{1f}$$

where $\Omega = (0, 1)$, $\Omega^- = (0, \alpha)$, $\Omega^+ = (\alpha, 1)$ and $\alpha \in \Omega$ is the interface position of the two materials. In the discussion from now on, we let $\rho_{min} \coloneqq min\{\rho^-, \rho^+\}$, $\rho_{max} \coloneqq max\{\rho^-, \rho^+\}$ and $\beta_{max} \coloneqq max\{\beta^-, \beta^+\}$, $\beta_{min} \coloneqq min\{\beta^-, \beta^+\}$.

IFE methods are desirable for solving interface problems with a mesh independent of the discontinuity of the coefficients associated with the material interfaces in the differential equations. The author of [8] introduced an IFE method for solving an interface problem of a two point boundary value problem. Afterwards, authors of [21, 11, 9, 2, 6, 15, 22, 7, 18, 14, 16, 1, 5, 10, 17, 19] developed IFE methods for solving elliptic interface problems, some time-dependent interface problems, Stokes interface problems as well as elasticity interface problems and so on. In particular, a Hermite cubic IFE space was developed in [13, 23] for solving interface problems of the 4-th order differential equations modeling a static Euler-Bernoulli beam and numerical examples were provided in those articles to show the optimal convergence of the related IFE method. A recent followup article [12] carried out an error analysis proving the optimal approximation capability for the Hermite cubic IFE space developed in [13, 23] and the optimal convergence of the numerical solution for the static Euler-Bernoulli beam produced in this IFE space by the usual Galerkin finite element scheme. However, so far there has been no error analysis for the IFE method developed in [23] to solve the time-dependent Euler-Bernoulli Beam interface problem, and this promotes us in this article to extend the error analysis reported in [12] to this fully discrete IFE method.

In the error analysis to be presented later, the standard Sobolev space defined on an open set $D \subseteq \Omega$ will be used: for every integer $m \geq 0$,

$$H^m(D) = \{w(x) \mid w^{(j)} \in L^2(D),\ j = 0, 1, \cdots, m\}, \tag{2}$$

on which we have the following norm and semi-norm:

$$\|w\|_{H^m(D)} = \sqrt{\sum_{j=0}^{m} \left\|w^{(j)}\right\|^2_{L^2(D)}},\ |w|_{H^m(D)} = \left\|w^{(m)}\right\|_{L^2(D)},\ \forall w \in H^m(D). \tag{3}$$

Also, we will use the following related Sobolev space: for every integer $m \geq 1$,

$$H_0^m(D) = \{w(x) \in H^m(D) \mid w^{(j)}|_{\partial D} = 0,\ j = 0, 1, \cdots, m-1\}. \tag{4}$$

In the case when $\alpha \in D$, we let $D^\pm = D \cap \Omega^\pm$ and we will consider the following space:

$$\tilde{H}^m(D) = \{w(x) \mid w|_{D^\pm} \in H^m(D^\pm)\}, \tag{5}$$

which is endowed with the following norm and semi-norm:

$$\begin{cases} \|w(x)\|_{\tilde{H}^m(D)} &= \sqrt{\|w\|^2_{H^m(D^-)} + \|w\|^2_{H^m(D^+)}} \\ |w(x)|_{\tilde{H}^m(D)} &= \sqrt{|w|^2_{H^m(D^-)} + |w|^2_{H^m(D^+)}} \end{cases} \quad \forall w \in \tilde{H}^m(D). \tag{6}$$