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CONVERGENCE ANALYSIS OF FINITE ELEMENT APPROXIMATION FOR 3-D MAGNETO-HEATING COUPLING MODEL

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Abstract. In this paper, the magneto-heating model is considered, where the nonlinear terms conclude the coupling magnetic diffusivity, the turbulent convection zone, the flow fields, ohmic heat, and α-quench. The highlights of this paper is consist of three parts. Firstly, the solvability of the model is derived from Rothe's method and Arzela-Ascoli theorem after setting up the decoupled semi-discrete system. Secondly, the well-posedness for the full-discrete scheme is arrived and the convergence order $O(h^{\min\{s,m\}} + \tau)$ is obtained, respectively, where the approximation scheme is based on backward Euler discretization in time and Nédélec-Lagrangian finite elements in space. At last, a numerical experiment demonstrates the expected convergence.

Key words. Magneto-heating model, finite element methods, nonlinear, solvability, convergent analysis.

1. Introduction

The phenomenon of magneto-heating has been achieved the main point of interest for many researches[19, 20, 30]. In [19], a magneto-heating model was established and the authors verified the well-posedness of the weak formulation by using the so-called regularity technique. In [24], the authors developed a mathematical model for magnetohydrodynamic flow of biofluids. The main objective was to explore the developmental performance of peristaltic transport with different zeta potentials in conjunction with magnetohydrodynamics and electrodynamics. In [13], the authors were committed to studying the convection flow of an electrically conducting and viscous incompressible fluids through isothermal vertical surfaces with Joule heating, when there exists a uniform transverse magnetic field fixed relative to the surface. Bermúdez and his cooperators studied the coupling of the equations of steady-state magnetohydrodynamics with the power equation when the buoyancy effect is considered in [3]. They showed two models and proved the existence of weak solutions. In [6], the authors researched a coupled system of Maxwell's equations with nonlinear heat equation while they employed the Rothe's method to prove the existence of the weak solutions for this coupled system.

There are many methods to prove the existence of solution for nonlinear equation [7, 10, 21, 23, 31, 37]. Rothe's method presents a first good insight into the structure of the solution of the investigated evolution problem. The method introduced by E. Rothe in 1930 [15]. It relies on the discretization in time and some energy estimates [6]. After then it can be further proved that the discrete solution is convergent to the solution of the original problem. Different from some other abstract methods for confirming the truth of existence, Rothe's method has a strong numerical aspect [15].

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The accurate prediction of magneto-heating phenomena is critical, especially to the basic understanding of the physical principles of controlling the electrodynamics and thermal behavior of the materials in these processing systems [17]. For these purposes, to look for a way to solve such a numerical problem is urgently needed, particularly with the strongly nonlinear conditions. Studies on the finite difference methods and finite volume methods had been applied to the magneto-thermal problems [11, 12, 28, 29]. Meanwhile, finite element method is another important approach for simulating these models due to its superior ability in handling problems that involve complex geometries [1, 2, 33]. It is specially powerful for nonlinear models. In [27], the author studied a nonlinear eddy current model and designed a nonlinear time semi-discrete numerical scheme. Then the Minty-Browder Theorem and a generalization of the div-curl lemma from the steady-state to the transient case were adopted to prove the convergence. As a result, the error estimates were achieved in time. In [5], for stellar magnetic activities, the authors proved the well-posedness of the dynamo system governed by a set of nonlinear PDEs with discontinuous physical coefficients in spherical geometry. Furthermore, they presented a full-discrete finite element approximation to the dynamo system and explored its convergence and stability. In [16], the main purpose was to prove an improved error estimate with $O(\tau + h^{\min\{1,\alpha\}})(\alpha > 0)$ for both time and space discretization than that in [9] for Maxwell's equations with a power-law nonlinear conductivity.

In this paper, compared with models mentioned above, the most significant differences of our model which is proposed in [34] can be summed into three points:

- The model is coupled with the turbulent convection zone and the flow fields.
- The nonlinear term concludes α-quench [5, 25].
- The coefficient of magnetic diffusion is temperature-dependent.

In order to get the existence of the weak solutions, we employ the Rothe's method. Firstly, the monotone theory is utilized to verify the unique solutions of time-discrete weak formulations. Then, by using the weak convergence theorem and Arzela-Ascoli theorem, we obtain that the time-discrete solutions of the magneto-heating coupling model converge to the solutions of the weak formulations. Next, we set up the full-discrete decoupled schemes by backward Euler discretization in time and Nédélec-Lagrangian elements in space. Furthermore, after the preparatory work, we obtain the convergence with the rates $O(h^{\min\{s,m\}} + \tau)$, where an a-prior L^∞ assumption of numerical solution is derived. At last, a simple numerical example is designed.

An outline of this paper is as follows. In section 2, we present the detailed information for the model and denotes some notations which will be used frequently in the rest of the paper. In Section 3, we employ time discretization based on Rothe's method to verify the solvability of the weak solutions for the problem (see Theorem 3.1). In Section 4, we construct the full-discrete scheme. Then based on interpolation theorem and the approximation properties between interpolations and finite element solutions, we obtain the error estimates (see Theorem 4.1), where an a-prior L^∞ assumption has to be inserted since the numerical scheme is the explicit decoupled. In Section 5, a numerical experiment is presented to verify theoretical results. Finally, some concluding remarks are given in the last section.

2. The magneto-heating coupling model and some notations

2.1. The model problem. A 3-D model is described by the governing equations [34]

$$\boldsymbol{B}_t+\nabla \times (\lambda(\theta)\nabla \times \boldsymbol{B}) = R_\alpha \nabla \times \left(\frac{f(\boldsymbol{x},t)\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}\right) + \nabla \times (\boldsymbol{U} \times \boldsymbol{B}), \ (0,T] \times \Omega, \tag{1}$$

$$\theta_t-\nabla \cdot (\kappa \nabla \theta) = \sigma(\theta)\Bigg(|\nabla \times \boldsymbol{B}|^2 - \nabla \times \boldsymbol{B} \cdot (\boldsymbol{U} \times \boldsymbol{B})$$

$$-R_\alpha \nabla \times \boldsymbol{B} \cdot \left(\frac{f(\boldsymbol{x},t)\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}\right)\Bigg), \qquad (0,T] \times \Omega, \tag{2}$$

where Ω is a bounded, convex, connected and Lipschtiz domain in R^3. $\boldsymbol{B}$ and θ mean the magnetic field and temperature, respectively. $f(\boldsymbol{x},t)$ and $\boldsymbol{U}$ are a model-oriented function and velocity of the fluid, respectively. R_α is a dynamo parameter. λ is the effective magnetic diffusivity which is also effected by the temperature. κ, γ are the thermal conductivity and a constant parameter, respectively. $\frac{R_\alpha f(\boldsymbol{x},t)}{1+\gamma|\boldsymbol{B}|^2}$ is called α-quench in [5, 25]. In some industrial experiments, the electric conductivity σ strongly depends on the temperature field such that $\sigma = \dfrac{b_1(\boldsymbol{x})}{(b_2(\boldsymbol{x})+b_3(\boldsymbol{x})\theta)^p}$ with $p>1$ and $\sigma = c_1(\boldsymbol{x})e^{-c_2(\boldsymbol{x})\theta}$(see [8, 35]), where $b_1(\boldsymbol{x}), b_2(\boldsymbol{x}), b_3(\boldsymbol{x}), c_1(\boldsymbol{x})$ and $c_2(\boldsymbol{x})$ are positive functions of space variables.

The equation (1) is equipped with the boundary condition

$$\boldsymbol{n} \times \boldsymbol{B} = 0, \quad \text{on} \quad \partial\Omega, \tag{3}$$

and the initial data

$$\boldsymbol{B}(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x}), \tag{4}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, $\boldsymbol{B}_0(\boldsymbol{x})$ is a given function. The equation (2) is equipped with the boundary condition

$$\theta = \theta_0, \qquad \text{on } (0,T] \times \Gamma_1, \tag{5}$$

$$-\kappa \frac{\partial \theta}{\partial \boldsymbol{n}} = 0, \qquad \text{on} \quad (0,T] \times \Gamma_2, \tag{6}$$

and the initial data

$$\theta(\boldsymbol{x},0) = \theta_0(\boldsymbol{x}), \tag{7}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Furthermore, in the initial condition, $\theta_0 \in L^\infty(\Omega)$ is the background temperature.

Moreover, we assume

$$\theta_0 \geq \theta_{min}, \quad |\sigma(\boldsymbol{x})| \quad \leq \quad \sigma_M, \tag{8}$$

where θ_{min}, σ_M are positive constants. We also assume that there exist constants λ_m, λ_M, κ_m, κ_M, f_M, and u_M such that

$$\begin{aligned} &0 < \lambda_m \leq \lambda(\boldsymbol{x}) \leq \lambda_M, \quad |f(\boldsymbol{x},t)|, |f_t(\boldsymbol{x},t)| \leq f_M, \\ &0 < \kappa_m \leq \kappa \leq \kappa_M, \qquad |\boldsymbol{U}(\boldsymbol{x},t)|, |\boldsymbol{U}_t(\boldsymbol{x},t)| \leq u_M, \end{aligned} \tag{9}$$

and λ, σ are two global Lipschitz continuous functions. For convenience, we define

$$q(\xi) := \sigma(\theta) = \sigma(\xi + \theta_0), \quad \nu(\xi) := \lambda(\theta) = \lambda(\xi+\theta_0),$$

$$K(\boldsymbol{B}) := |\nabla \times \boldsymbol{B}|^2 - \nabla \times \boldsymbol{B} \cdot (\boldsymbol{U} \times \boldsymbol{B}) - R_\alpha \nabla \times \boldsymbol{B} \cdot \left(\frac{f(\boldsymbol{x},t)\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}\right),$$

$Q_T = (0, T] \times \Omega.$

2.2. Notations. Firstly, we introduce some function spaces and notations which will be used throughout the paper. Here $\boldsymbol{W}^{\alpha,p}(\Omega) = (W^{\alpha,p}(\Omega))^3$ means the standard Sobolev vector-valued functions space with norm $\|\cdot\|_{\alpha,p}$ in three dimension. When $p = 2$, we denote the space $\boldsymbol{W}^{\alpha,2}(\Omega) = \boldsymbol{H}^{\alpha}(\Omega) = (H^{\alpha}(\Omega))^3$ with norm $\|\cdot\|_{\alpha}$. When $\alpha = 0$, the space $\boldsymbol{H}^0(\Omega)$ coincides with $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^3$ equipped with norm $\|\cdot\|_0$. For simplify, we sometimes note the norm for $\|\cdot\|$ in the absence of confusion. For a time-dependent function $\boldsymbol{u}(\boldsymbol{x}, t)$, the Bochner space is involved [14]

$$L^q(0, T; \boldsymbol{H}^{\alpha}(\Omega)) = \left\{\boldsymbol{u} : (0, T) \to \boldsymbol{H}^{\alpha}(\Omega);\ \|\boldsymbol{u}\|_{L^q(0,T;\boldsymbol{H}^{\alpha}(\Omega))} < \infty\right\},$$

where

$$\|\boldsymbol{u}\|_{L^q(0,T;\boldsymbol{H}^{\alpha}(\Omega))} = \begin{cases} \left(\displaystyle\int_0^T \|\boldsymbol{u}(\cdot, t)\|_{\alpha}^q \mathrm{d}t\right)^{1/q}, & 1 \le q < \infty, \\ \displaystyle\max_{0\le t\le T} \|\boldsymbol{u}(\cdot, t)\|_{\alpha}, & q = \infty. \end{cases}$$

Now we show some other commonly notations:

$$H(\mathrm{curl}; \Omega) := \left\{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\ : \nabla \times \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\right\},$$

$$V := H_0(\mathrm{curl}; \Omega) = \{\boldsymbol{u} \in H(\mathrm{curl}; \Omega)\ : \boldsymbol{n} \times \boldsymbol{u} = 0 \ \text{ on } \partial\Omega\}.$$

We also need define the functional space for the heat equation

$$Y := H_0^1(\Omega) = \left\{v \in H^1(\Omega), v|_{\Gamma_1} = 0\right\}.$$

We introduce the cut-off function $\mathcal{C}_r$ to deal with the nonlinear term of (2)

$$\mathcal{C}_r(\boldsymbol{x}) = \begin{cases} r & \text{if } x > r, \\ x & \text{if } |x| \le r, \\ -r & \text{if } x < -r, \end{cases}$$

where r is a positive constant. Then we can get the truncated form of equation (2)

$$\theta_t - \nabla \cdot (\kappa\nabla\theta) = \mathcal{C}_r(\sigma(\theta)K(\boldsymbol{B})), \quad (0, T] \times \Omega. \tag{10}$$

From now on, we analysis the truncated system.

Throughout this paper, we shall frequently use C and C_r to denote a generic constant, while C_r depends on the cutoff constant r.

3. Solvability of the solutions for the model

The coupling system (1), (3)-(7), (10) can be equivalent to the following variational problem: For the given initial data $\boldsymbol{B}_0$, θ_0, $\xi_0 = 0$ and for any $t \in (0, T]$, find $\boldsymbol{B} \in V$ and $\xi \in Y$ such that

$$(\boldsymbol{B}_t, \boldsymbol{\Phi}) + (\nu(\xi)\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{\Phi}) = R_{\alpha}\left(\frac{f(\boldsymbol{x}, t)\boldsymbol{B}}{1 + \gamma|\boldsymbol{B}|^2}, \nabla \times \boldsymbol{\Phi}\right) + (\boldsymbol{U} \times \boldsymbol{B}, \nabla \times \boldsymbol{\Phi}),\ \forall\ \boldsymbol{\Phi} \in V, \tag{11}$$

$$(\xi_t, \Upsilon) + (\kappa\nabla\xi, \nabla\Upsilon) = (\mathcal{C}_r(q(\xi)K(\boldsymbol{B})), \Upsilon) - (\kappa\nabla\theta_0, \nabla\Upsilon), \forall\ \Upsilon \in Y. \tag{12}$$

In this section, we use the Rothe's method to prove the solvability of the problem (11)-(12). We take a fixed time step τ and split the time interval into n parts, i.e. $T = n\tau$, where n is a positive integer. Denote

$$t_k = \tau k, \quad w^k = w(t_k), \quad \delta_\tau w^k = \frac{w^k - w^{k-1}}{\tau},$$

then we have a time discretized form of weak formulations (11)-(12) as follows

$$
\begin{aligned}
&\left(\delta_\tau \boldsymbol{B}^k, \boldsymbol{\Phi}\right) + \left(\nu\left(\xi^{k-1}\right)\nabla\times\boldsymbol{B}^k, \nabla\times\boldsymbol{\Phi}\right) = R_\alpha\left(\frac{f(\boldsymbol{x},t_k)\boldsymbol{B}^k}{1+\gamma|\boldsymbol{B}^k|^2}, \nabla\times\boldsymbol{\Phi}\right)\\
&\qquad + \left(\boldsymbol{U}^k\times\boldsymbol{B}^k, \nabla\times\boldsymbol{\Phi}\right),\ \forall\ \boldsymbol{\Phi}\in V, \qquad (13)
\end{aligned}
$$

$$
\left(\delta_\tau \xi^k, \Upsilon\right) + \left(\kappa\nabla\xi^k, \nabla\Upsilon\right) = \left(\mathcal{C}_r(q(\xi^{k-1})K(\boldsymbol{B}^k)), \Upsilon\right) - \left(\kappa\nabla\theta_0, \nabla\Upsilon\right), \forall\ \Upsilon\in Y. \qquad (14)
$$

Lemma 3.1. For any $k = 1, \cdots, n$, there exists a unique $\boldsymbol{B}^k \in V,\ \xi^k \in Y$ to solve (13)-(14).

Proof. Firstly, we define the operator $M_\lambda : V \to V^*$, where V^* means its dual space.

$$
\begin{aligned}
\langle M_\lambda(\boldsymbol{B}), \boldsymbol{\Phi}\rangle = &\frac{1}{\tau}\left(\boldsymbol{B}, \boldsymbol{\Phi}\right) + \left(\nu\nabla\times\boldsymbol{B}, \nabla\times\boldsymbol{\Phi}\right) - R_\alpha\left(\frac{f(\boldsymbol{x},t)\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}, \nabla\times\boldsymbol{\Phi}\right)\\
&- \left(\boldsymbol{U}\times\boldsymbol{B}, \nabla\times\boldsymbol{\Phi}\right).
\end{aligned}
$$

Next, we verify that the operator is bounded, monotone, coercive and hemi-continuous. In particular, the hemicontinuity is obviously established.

We first show the boundedness as follows

$$
\begin{aligned}
\langle M_\lambda(\boldsymbol{B}), \boldsymbol{\Phi}\rangle \quad &\leq \quad \frac{1}{\tau}\|\boldsymbol{B}\|\|\boldsymbol{\Phi}\| + \lambda_M\|\nabla\times\boldsymbol{B}\|\|\nabla\times\boldsymbol{\Phi}\| + R_\alpha f_M\|\boldsymbol{B}\|\|\nabla\times\boldsymbol{\Phi}\|\\
&+ \quad u_M\|\boldsymbol{B}\|\|\nabla\times\boldsymbol{\Phi}\| \leq C\|\boldsymbol{B}\|_V\|\boldsymbol{\Phi}\|_V, \qquad (15)
\end{aligned}
$$

where $C = \max\{1/\tau, \lambda_M, R_\alpha f_M, u_M\}$. Then we have $\|M_\lambda(\boldsymbol{B})\|_{V^*} \leq C\|\boldsymbol{B}\|_V,\ \forall\ \boldsymbol{B}\in V$.

Now, we verify the monotonicity, for any $\boldsymbol{B}_1, \boldsymbol{B}_2 \in V$,

$$
\begin{aligned}
\langle M_\lambda(\boldsymbol{B}_1) - M_\lambda(\boldsymbol{B}_2), \boldsymbol{B}_1-\boldsymbol{B}_2\rangle = &\frac{1}{\tau}\left(\boldsymbol{B}_1-\boldsymbol{B}_2, \boldsymbol{B}_1-\boldsymbol{B}_2\right) + \left(\nu\left(\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\right),\right.\\
&\left.\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\right) - R_\alpha\left(\frac{f(\boldsymbol{x},t)\boldsymbol{B}_1}{1+\gamma|\boldsymbol{B}_1|^2} - \frac{f(\boldsymbol{x},t)\boldsymbol{B}_2}{1+\gamma|\boldsymbol{B}_2|^2}, \nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\right)\\
&- \left(\boldsymbol{U}\times\boldsymbol{B}_1 - \boldsymbol{U}\times\boldsymbol{B}_2, \nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\right)\\
&= \sum_{i=1}^{4} I_i.
\end{aligned}
$$

Then, by Cauchy's inequality, Young's inequality and Lemma 2.3 in [34], we have

$$
\begin{aligned}
|I_1| \quad &= \quad \frac{1}{\tau}\|\boldsymbol{B}_1-\boldsymbol{B}_2\|^2,\\
|I_2| \quad &\geq \quad \lambda_m\|\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\|^2,\\
|I_3| \quad &\leq \quad \frac{9}{4}R_\alpha f_M\|\boldsymbol{B}_1-\boldsymbol{B}_2\|\|\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\|\\
&\leq \quad \frac{9}{4}R_\alpha f_M\left(\frac{1}{4\epsilon_1}\|\boldsymbol{B}_1-\boldsymbol{B}_2\|^2 + \epsilon_1\|\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\|^2\right),\\
|I_4| \quad &\leq \quad u_M\|\boldsymbol{B}_1-\boldsymbol{B}_2\|\|\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\|\\
&\leq \quad u_M\left(\frac{1}{4\epsilon_2}\|\boldsymbol{B}_1-\boldsymbol{B}_2\|^2 + \epsilon_2\|\nabla\times\boldsymbol{B}_1-\nabla\times\boldsymbol{B}_2\|^2\right).
\end{aligned}
$$

By choosing the proper parameters $\tau, \epsilon_1, \epsilon_2 > 0$ such that

$$
C_1 = \frac{1}{\tau} - \frac{9R_\alpha f_M}{16\epsilon_1} - \frac{u_M}{4\epsilon_2} > 0, \quad C_2 = \lambda_m - \frac{9\epsilon_1}{4}R_\alpha f_M - u_M\epsilon_2 > 0,
$$

we have

$$\begin{aligned}&\langle M_\lambda(\boldsymbol{B}_1) - M_\lambda(\boldsymbol{B}_2), \boldsymbol{B}_1 - \boldsymbol{B}_2\rangle \\ &\geq \min\{C_1, C_2\}\left(\|\boldsymbol{B}_1 - \boldsymbol{B}_2\|^2 + \|\nabla\times\boldsymbol{B}_1 - \nabla\times\boldsymbol{B}_2\|^2\right) \geq 0.\end{aligned}$$

Next we show the coercive of the operator M_λ.

$$\begin{aligned}\langle M_\lambda(\boldsymbol{B}), \boldsymbol{B}\rangle &\geq \frac{1}{\tau}\|\boldsymbol{B}\|^2 + \lambda_m\|\nabla\times\boldsymbol{B}\|^2 - \left(\frac{R_\alpha f_M}{4\epsilon_3}\|\boldsymbol{B}\|^2 + \epsilon_3 R_\alpha f_M\|\nabla\times\boldsymbol{B}\|^2\right) \\ &\quad - \left(\frac{u_M}{4\epsilon_4}\|\boldsymbol{B}\|^2 + \epsilon_4 u_M\|\nabla\times\boldsymbol{B}\|^2\right) \\ &\geq C_3\|\boldsymbol{B}\|_V^2,\end{aligned}$$

by choosing the parameters $\tau, \epsilon_3, \epsilon_4 > 0$ such that

$$C_3 = \min\left\{\frac{1}{\tau} - \frac{R_\alpha f_M}{4\epsilon_3} - \frac{u_M}{4\epsilon_4}, \lambda_m - \epsilon_3 R_\alpha f_M - \epsilon_4 u_M\right\} > 0.$$

As mentioned above, we have proved these properties. Assume that k is given and $\boldsymbol{B}^{k-1}, \xi^{k-1}$ are known, for any $\boldsymbol{\Phi} \in V$, then the operator equation

$$\left\langle M_\lambda(\boldsymbol{B}^k), \boldsymbol{\Phi}\right\rangle = \frac{1}{\tau}\left(\boldsymbol{B}^{k-1}, \boldsymbol{\Phi}\right) \tag{16}$$

has a solution $\boldsymbol{B}^k \in V$ [36]. From Theorem 6.1 [32] and Lemma 6.1.1 [26], we can obtain that the solution of the equation (13) is unique. The existence and uniqueness of the solution of the equation (14) is trivial based on Lax-Milgram lemma since it is a linear problem after we know $\mathbf{B}^k$ and ξ^{k-1}. ■

Now, we show some boundedness for $\boldsymbol{B}^k$ in the next lemma.

Lemma 3.2. Suppose that $\boldsymbol{B}^k$ is the solution of (13)-(14). Then there holds

$$\max_{1\leq k\leq n}\|\boldsymbol{B}^k\|^2 + \tau\lambda_m\sum_{k=1}^{n}\|\nabla\times\boldsymbol{B}^k\|^2 \leq C\|\boldsymbol{B}_0\|^2, \tag{17}$$

$$\sum_{k=1}^{n}\tau\|\delta_\tau\boldsymbol{B}^k\|^2_{H^{-1}(\mathrm{curl};\Omega)} \leq C, \tag{18}$$

where C is a positive constant and independent of τ. $H^{-1}(\mathrm{curl};\Omega)$ is the dual space of $H_0(\mathrm{curl};\Omega)$.

The bounded estimates for ξ^k are presented in the next lemma.

Lemma 3.3. There exists a positive constant C_r, which depends on the parameter r of the cut-off function $\mathcal{C}_r$ and independent of τ, such that

$$\max_{1\leq k\leq n}\|\xi^k\|^2 + \sum_{k=1}^{n}\kappa\tau\|\nabla\xi^k\|^2 \leq C_r, \tag{19}$$

$$\sum_{k=1}^{n}\|\delta_\tau\xi^k\|^2\tau + \frac{\kappa_m}{2}\left[\|\nabla\xi^n\|^2 + \sum_{k=1}^{n}\tau\|\nabla\xi^k - \nabla\xi^{k-1}\|^2\right] \leq C_r + \kappa_M\|\nabla\xi^0\|^2, \tag{20}$$

$$\max_{1\leq k\leq n}\|\delta_\tau\xi^k\|_{H^{-1}(\Omega)} \leq C_r, \tag{21}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

The proof of Lemma 3.2 and Lemma 3.3 is trivial, so we omit it.

Before we show the first main theorem in this section, we need to construct the piecewise-constant and piecewise-linear functions in time, i.e., $\forall t \in (t_{k-1}, t_k]$,

$$\begin{aligned}
&\bar{\boldsymbol{B}}^n(0) = \boldsymbol{B}(0), && \bar{\xi}^n(0) = \xi_0,\\
&\bar{\boldsymbol{B}}^n(t) = \boldsymbol{B}^k, && \bar{\xi}^n(t) = \xi^k,\\
&\boldsymbol{B}^n(t) = \boldsymbol{B}^{k-1} + (t - t_{k-1})\delta_\tau \boldsymbol{B}^k, && \xi^n(t) = \xi^{k-1} + (t - t_{k-1})\delta_\tau \xi^k,\\
&\bar{\nu}^n(0) = \nu(\xi_0), && \bar{q}^n(0) = q(\xi_0),\\
&\bar{\nu}^n(t) = \nu(\xi^k), && \bar{q}^n(t) = q(\xi^k),\\
&\bar{f}^n(t) = f(t_k), && \bar{\boldsymbol{U}}^n(t) = \boldsymbol{U}(t_k).
\end{aligned}$$

Now using the above notations, the equations (13)-(14) can be rewritten as

$$\begin{aligned}
(\partial_t \boldsymbol{B}^n, \boldsymbol{\Phi}) + \left(\bar{\nu}^n\left(t-\tau\right) \nabla \times \bar{\boldsymbol{B}}^n, \nabla \times \boldsymbol{\Phi}\right) &= R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \boldsymbol{\Phi}\right)\\
&\quad + \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \boldsymbol{\Phi}\right), \quad \forall\, \boldsymbol{\Phi} \in V, \qquad (22)
\end{aligned}$$

$$\begin{aligned}
(\partial_t \xi^n, \Upsilon) + \left(\kappa \nabla \bar{\xi}^n, \nabla \Upsilon\right) &= \left(\mathcal{C}_r\left(\bar{q}^n\left(t-\tau\right) K\left(\bar{\boldsymbol{B}}^n\right)\right), \Upsilon\right)\\
&\quad - \left(\kappa \nabla \theta_0, \nabla \Upsilon\right), \qquad \forall\ \Upsilon \in Y. \qquad (23)
\end{aligned}$$

Theorem 3.1. *Assume that f is Lipschitz continuous in time, then there exist $\boldsymbol{B}$ and ξ to solve (13) - (14).*

Proof. The proof is divided into three parts.

(I) Owing to Lemma 3.3, we have

$$\int_0^t \|\partial_t \xi^n\|^2 \mathrm{d}t + \max_{t\in[0,T]} \|\bar{\xi}^n\|^2_{H^1(\Omega)} \leq C_r.$$

Then there exists a $\xi \in C\left([0,T]; L^2\left(\Omega\right)\right) \cap L^\infty\left([0,T]\,; H^1\left(\Omega\right)\right)$ with $\partial_t \xi \in L^2\big(\left(0,T\right); L^2\left(\Omega\right)\big)$ such that [15]

$$\xi^n \to \xi \ \text{ in } C\left([0,T]; L^2\left(\Omega\right)\right) \ \text{ and } \ \bar{\xi}^n \to \xi \text{ in } L^2\left(\left(0,T\right); L^2\left(\Omega\right)\right), \qquad (24)$$

$$\bar{\xi}^n(t) \rightharpoonup \xi(t) \text{ in } H^1_0(\Omega), \quad \forall t \in [0,T]. \qquad (25)$$

Based on (24) and (25), we arrive at $\xi^n \to \xi$ and $\bar{\xi}^n \to \xi$ a.e. in Q_T. From the Lipschitz continuity of ν, q, we have

$$\begin{aligned}
&\nu(\xi^n) \to \nu(\xi), \quad \nu(\bar{\xi}^n) \to \nu(\xi) \ \text{ a.e. in } Q_T,\\
&q(\xi^n) \to q(\xi), \quad q(\bar{\xi}^n) \to q(\xi) \ \text{ a.e. in } Q_T.
\end{aligned}$$

Now we show that $\bar{\nu}^n(t-\tau)$, $\bar{\nu}^n(t)$ and $\bar{q}^n(t-\tau)$, $\bar{q}^n(t)$ have the same limit in $L^2((0,T); L^2(\Omega))$, respectively. For $\bar{\nu}^n(t-\tau)$ and $\bar{\nu}^n(t)$, from (20), we have

$$\begin{aligned}
\int_0^T \|\bar{\nu}^n(t-\tau) - \bar{\nu}^n(t)\|^2 \mathrm{d}t &= \sum_{k=1}^n \|\nu(\xi^{k-1}) - \nu(\xi^k)\|^2 \tau\\
\leq C \sum_{k=1}^n \|\xi^{k-1} - \xi^k\|^2 \tau &\leq C\tau^2 \sum_{k=1}^n \|\delta \xi^k\|^2 \tau \leq C_r \tau^2,
\end{aligned}$$

which leads to

$$\lim_{n\to\infty} \int_0^T \|\bar{\nu}^n(t-\tau) - \bar{\nu}^n(t)\|^2 \mathrm{d}t = 0.$$

Then by using triangle inequality, we can reach

$$\lim_{n\to\infty}\int_0^T \|\bar{\nu}^n(t-\tau)-\nu(\xi(t))\|^2 \mathrm{d}t = 0.$$

Similarly, we have

$$\lim_{n\to\infty}\int_0^T \|\bar{q}^n(t-\tau)-q(\xi(t))\|^2 \mathrm{d}t = 0.$$

Therefore, there holds

$$\begin{aligned}&\bar{\nu}^n(t)\to\nu(\xi),\ \bar{\nu}^n(t-\tau)\to\nu(\xi),\\ &\bar{q}^n(t)\to q(\xi),\ \bar{q}^n(t-\tau)\to q(\xi), \qquad \text{in } L^2((0,T);L^2(\Omega)).\end{aligned} \tag{26}$$

(Ⅱ)According to Lemma 3.2, we get

$$\left|(\bar{\boldsymbol{B}}^n-\boldsymbol{B}^n,\boldsymbol{\Phi})\right| \le \tau|(\partial_t\boldsymbol{B}^n,\boldsymbol{\Phi})| \le \tau\|\partial_t\boldsymbol{B}^n\|_{H^{-1}(\mathrm{curl};\Omega)}\|\boldsymbol{\Phi}\|_{H(\mathrm{curl};\Omega)},$$

which means

$$\begin{aligned}\|\bar{\boldsymbol{B}}^n-\boldsymbol{B}^n\|_{L^2((0,T);H^{-1}(\mathrm{curl};\Omega))} &\le \tau\int_0^T \|\partial_t\boldsymbol{B}^n\|^2_{H^{-1}(\mathrm{curl};\Omega)}\mathrm{d}t\\ &\le C\tau\to 0,\quad n\to\infty.\end{aligned} \tag{27}$$

From

$$\begin{aligned}\int_0^T\|\boldsymbol{B}^n\|^2\mathrm{d}t &= \sum_{k=1}^n\int_{t_{k-1}}^{t_k}\|\boldsymbol{B}^{k-1}+(t-t_{k-1})\delta\boldsymbol{B}^k\|^2\mathrm{d}t\\ &\le \sum_{k=1}^n(\|\boldsymbol{B}^{k-1}\|^2+\|\boldsymbol{B}^k-\boldsymbol{B}^{k-1}\|^2)\tau\\ &\le \|\boldsymbol{B}_0\|^2 + C\sum_{k=1}^n\|\boldsymbol{B}^k\|^2\tau\le C,\\ \int_0^T\|\bar{\boldsymbol{B}}^n\|^2\mathrm{d}t &= \sum_{k=1}^n\|\boldsymbol{B}^k\|^2\tau\le C,\end{aligned}$$

and Lemma 3.2, we have

$$\begin{aligned}&\int_0^T(\|\bar{\boldsymbol{B}}^n\|^2+\|\nabla\times\bar{\boldsymbol{B}}^n\|^2)\mathrm{d}t\le C,\\ &\int_0^T(\|\boldsymbol{B}^n\|^2+\|\partial_t\boldsymbol{B}^n\|^2_{H^{-1}(\mathrm{curl};\Omega)})\mathrm{d}t\le C.\end{aligned}$$

Moreover, $L^2\left((0,T);\boldsymbol{L}^2(\Omega)\right)$ is reflexive Banach space, which implies that there exists subsequences of $\bar{\boldsymbol{B}}^n,\boldsymbol{B}^n$ (we also denote $\bar{\boldsymbol{B}}^n,\boldsymbol{B}^n$) and $\boldsymbol{B}_1,\boldsymbol{B}$ such that $\bar{\boldsymbol{B}}^n\rightharpoonup \boldsymbol{B}_1,\ \boldsymbol{B}^n\rightharpoonup\boldsymbol{B}$, where $'\rightharpoonup'$ means the weak convergence. We also have [6, 27]

$$\lim_{n\to\infty}\int_0^T\left(\bar{\boldsymbol{B}}^n,\Xi\boldsymbol{B}^n\right)\mathrm{d}t=\int_0^T(\boldsymbol{B}_1,\Xi\boldsymbol{B})\,\mathrm{d}t,\quad \forall\ \Xi\in C_0^\infty\left(\bar{\Omega}\right).$$

Next, we will illustrate that $\bar{\boldsymbol{B}}^n\rightharpoonup\boldsymbol{B}$. For any $\boldsymbol{p}\in L^2((0,T);\boldsymbol{L}^2(\Omega))$, we define

$$0\le\lim_{n\to\infty}\int_0^T\left(\bar{\boldsymbol{B}}^n-\boldsymbol{p},\Xi\left(\bar{\boldsymbol{B}}^n-\boldsymbol{p}\right)\right)\mathrm{d}t := \lim_{n\to\infty}\sum_{k=1}^4(-1)^{i+1}II_i, \tag{28}$$

where $\Xi \in C_0^\infty(\bar{\Omega})$ is nonnegative, and

$$II_1 = \int_0^T \left(\bar{\boldsymbol{B}}^n, \Xi \bar{\boldsymbol{B}}^n\right) \mathrm{d}t = \int_0^T \left(\bar{\boldsymbol{B}}^n, \Xi\left(\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\right)\right) \mathrm{d}t + \int_0^T \left(\bar{\boldsymbol{B}}^n, \Xi \boldsymbol{B}^n\right) \mathrm{d}t.$$

Based on Lemma 3.2 and (27), we have

$$\begin{aligned}\int_0^T \left(\bar{\boldsymbol{B}}^n, \Xi\left(\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\right)\right) \mathrm{d}t &\leq C\|\bar{\boldsymbol{B}}^n\|_{L^2(0,T;H(\mathrm{curl};\Omega))}\|\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\|_{L^2(0,T;H^{-1}(\mathrm{curl};\Omega))} \\ &\leq C\|\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\|_{L^2(0,T;H^{-1}(\mathrm{curl};\Omega))} \to 0, \quad \text{if} \quad n \to \infty,\end{aligned}$$

which means

$$\lim_{n\to\infty} II_1 = \int_0^T (\boldsymbol{B}_1, \Xi \boldsymbol{B})\mathrm{d}t.$$

The space $L^2((0,T);\boldsymbol{C}^\infty(\Omega))$ is dense in $L^2((0,T);\boldsymbol{L}^2(\Omega))$. Then for any $\epsilon > 0$, there exists $\boldsymbol{p}_\epsilon \in L^2((0,T);\boldsymbol{C}^\infty(\Omega))$ such that $\|\boldsymbol{p} - \boldsymbol{p}_\epsilon\|_{L^2((0,T);\boldsymbol{L}^2(\Omega))} \leq \epsilon$. Hence,

$$\begin{aligned}II_2 &= \int_0^T \left(\boldsymbol{p}, \Xi \bar{\boldsymbol{B}}^n\right) \mathrm{d}t \\ &= \int_0^T \left(\boldsymbol{p}_\epsilon, \Xi\left(\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\right)\right) \mathrm{d}t + \int_0^T \left(\boldsymbol{p} - \boldsymbol{p}_\epsilon, \Xi\left(\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\right)\right) \mathrm{d}t + \int_0^T \left(\boldsymbol{p}, \Xi \boldsymbol{B}^n\right) \mathrm{d}t \\ &:= \sum_{i=1}^3 I_i,\end{aligned}$$

where

$$\begin{aligned}|I_1| &\leq C\|\boldsymbol{p}_\epsilon\|_{L^2((0,T);H(\mathrm{curl};\Omega))}\|\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\|_{L^2((0,T);H^{-1}(\mathrm{curl};\Omega))} \\ &\leq C_\epsilon\|\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\|_{L^2((0,T);H^{-1}(\mathrm{curl};\Omega))} \to 0, \quad n \to \infty, \\ |I_2| &\leq C\|\boldsymbol{p} - \boldsymbol{p}_\epsilon\|_{L^2((0,T);L^2(\Omega))}\|\bar{\boldsymbol{B}}^n - \boldsymbol{B}^n\|_{L^2((0,T);L^2(\Omega))} \leq C\epsilon \to 0, \quad \epsilon \to 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} II_2 &= \int_0^T (\boldsymbol{p}, \Xi \boldsymbol{B})\mathrm{d}t, \quad \lim_{n\to\infty} II_3 = \int_0^T (\boldsymbol{p}, \Xi \boldsymbol{p})\mathrm{d}t, \\ \lim_{n\to\infty} II_4 &= \lim_{n\to\infty} \int_0^T (\bar{\boldsymbol{B}}^n, \Xi \boldsymbol{p})\mathrm{d}t = \int_0^T (\boldsymbol{B}_1, \Xi \boldsymbol{p})\mathrm{d}t.\end{aligned}$$

We can see that

$$\lim_{n\to\infty} \int_0^T \left(\bar{\boldsymbol{B}}^n - \boldsymbol{p}, \Xi\left(\bar{\boldsymbol{B}}^n - \boldsymbol{p}\right)\right) \mathrm{d}t = \int_0^T \left(\boldsymbol{B}_1 - \boldsymbol{p}, \Xi\left(\boldsymbol{B} - \boldsymbol{p}\right)\right) \geq 0.$$

Now, setting $\epsilon > 0$ and $\boldsymbol{p} = \boldsymbol{B} + \epsilon \boldsymbol{v}, \boldsymbol{v} \in L^2((0,T);\boldsymbol{L}^2(\Omega))$, we have

$$\lim_{\epsilon\to 0} \int_0^T (\boldsymbol{B}_1 - \boldsymbol{B}, \Xi \boldsymbol{v}) \leq 0.$$

Replacing $\boldsymbol{v}$ with $-\boldsymbol{v}$ implies

$$\lim_{\epsilon\to 0} \int_0^T (\boldsymbol{B}_1 - \boldsymbol{B}, \Xi \boldsymbol{v}) \geq 0.$$

Therefore, we obtain

$$\lim_{\epsilon\to 0} \int_0^T (\boldsymbol{B}_1 - \boldsymbol{B}, \Xi \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in L^2((0,T);\boldsymbol{L}^2(\Omega)).$$

Hence, $\boldsymbol{B}_1 = \boldsymbol{B}$ a.e. in Q_T, i.e.,

$$\bar{\boldsymbol{B}}^n \rightharpoonup \boldsymbol{B}, \quad \text{in } L^2((0,T);\boldsymbol{L}^2(\Omega)). \tag{29}$$

Let $\boldsymbol{p} = \boldsymbol{B}$ in (28), we have

$$0 = \lim_{n\to\infty}\int_0^T (\bar{\boldsymbol{B}}^n - \boldsymbol{B}, \Xi(\bar{\boldsymbol{B}}^n - \boldsymbol{B}))\mathrm{d}t \geq \lim_{n\to\infty}\int_0^T (\Xi, |\bar{\boldsymbol{B}}^n - \boldsymbol{B}|^2) \geq 0, \quad \forall \Xi \in C_0^\infty(\bar{\Omega}),$$

which means

$$\bar{\boldsymbol{B}}^n \to \boldsymbol{B}, \quad \text{in } L^2((0,T);\boldsymbol{L}^2(\Omega)). \tag{30}$$

Setting $\boldsymbol{\Phi} \in H_0(\text{curl};\Omega)$, then we get

$$\begin{aligned}\lim_{n\to\infty}\int_0^T \left(\nabla\times\bar{\boldsymbol{B}}^n, \boldsymbol{\Phi}\right)\mathrm{d}t &= \lim_{n\to\infty}\int_0^T \left(\bar{\boldsymbol{B}}^n, \nabla\times\boldsymbol{\Phi}\right)\mathrm{d}t\\ &= \int_0^T (\boldsymbol{B}, \nabla\times\boldsymbol{\Phi})\,\mathrm{d}t = \int_0^T (\nabla\times\boldsymbol{B}, \boldsymbol{\Phi})\,\mathrm{d}t.\end{aligned} \tag{31}$$

It's known that $L^2((0,T);H(\text{curl};\Omega))$ is reflexive, and based on Lemma 3.2, there exists a $\boldsymbol{z} \in L^2((0,T);H(\text{curl};\Omega))$ such that

$$\int_0^t (\partial_t \boldsymbol{B}^n, \boldsymbol{\Phi})\,ds \to \int_0^t (\boldsymbol{z}, \boldsymbol{\Phi})\,ds, \quad n\to\infty,$$

and

$$\begin{aligned}(\boldsymbol{B}^n(t), \boldsymbol{\Phi}) - (\boldsymbol{B}^n(0), \boldsymbol{\Phi}) = \int_0^t (\partial_t \boldsymbol{B}^n, \boldsymbol{\Phi})\,ds &\leq \int_0^t \|\partial_t \boldsymbol{B}^n\|_{H^{-1}(\text{curl};\Omega)}\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)}ds\\ &\leq C\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)}.\end{aligned}$$

Therefore, we have

$$(\boldsymbol{B}^n(t), \boldsymbol{\Phi}) \leq C\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)} + \|\boldsymbol{B}_0\|_{H^{-1}(\text{curl};\Omega)}\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)} \leq C\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)},$$

which leads to

$$\|\boldsymbol{B}^n(t)\|_{H^{-1}(\text{curl};\Omega)} \leq C.$$

For any t_1, t_2, there holds

$$\begin{aligned}|(\boldsymbol{B}^n(t_1) - \boldsymbol{B}^n(t_2), \boldsymbol{\Phi})| \leq \left|\int_{t_1}^{t_2} (\partial_t \boldsymbol{B}^n, \boldsymbol{\Phi})ds\right| &\leq \int_{t_1}^{t_2} \|\partial_t \boldsymbol{B}^n\|_{H^{-1}(\text{curl};\Omega)}\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)}ds\\ &\leq \sqrt{\int_{t_1}^{t_2} 1^2 ds}\sqrt{\int_{t_1}^{t_2} \|\partial_t \boldsymbol{B}^n\|^2_{H^{-1}(\text{curl};\Omega)}ds}\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)}\\ &\leq C\sqrt{|t_1 - t_2|}\|\boldsymbol{\Phi}\|_{H(\text{curl};\Omega)},\end{aligned}$$

which implies

$$\|\boldsymbol{B}^n(t_1) - \boldsymbol{B}^n(t_2)\|_{H^{-1}(\text{curl};\Omega)} \leq C\sqrt{|t_1 - t_2|}.$$

Using the modification of Arzela-Ascoli theorem [6, 15] yields

$$\lim_{n\to\infty}(\boldsymbol{B}^n, \boldsymbol{\Phi}) = (\boldsymbol{B}, \boldsymbol{\Phi}), \quad \forall \boldsymbol{\Phi} \in H_0(\text{curl};\Omega), t \in [0,T].$$

Then, we can obtain $\boldsymbol{z} = \partial_t \boldsymbol{B}$ a.e. in Q_T by

$$\int_0^t (\partial_t \boldsymbol{B}, \boldsymbol{\Phi})ds \quad = \quad (\boldsymbol{B}(t) - \boldsymbol{B}_0, \boldsymbol{\Phi}) = \lim_{n\to\infty}(\boldsymbol{B}^n(t) - \boldsymbol{B}^n(0), \boldsymbol{\Phi})$$

$$= \lim_{n\to\infty}\int_0^t (\partial_t \boldsymbol{B}^n, \boldsymbol{\Phi})ds = \int_0^t (\boldsymbol{z}, \boldsymbol{\Phi})ds. \tag{32}$$

Owing to the Lipschitz continuity of f and U, we have

$$\|f(t_1) - f(t_2)\| \le C|t_1 - t_2|, \quad \forall t_1, t_2 \in [0, T],$$

which implies

$$\int_0^T \|\bar{f}^n - f\|^2 \mathrm{d}t = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|f(t_k) - f(t)\|^2 \mathrm{d}t \le C\tau^2, \quad n \to \infty. \tag{33}$$

Similarly, we have

$$\int_0^T \|\bar{U}^n - U\|^2 \mathrm{d}t = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|U(t_k) - U(t)\|^2 \mathrm{d}t \le C\tau^2, \quad n \to \infty. \tag{34}$$

Now, we have to verify $\nabla \times \bar{\boldsymbol{B}}^n \to \nabla \times \boldsymbol{B}$ in $L^2((0,T); \boldsymbol{L}^2(\Omega))$. From (30), by choosing $t \in [0,T]$, such that $\bar{\boldsymbol{B}}^n(t) \to \boldsymbol{B}(t)$ in $\boldsymbol{L}^2(\Omega)$, and assuming that $t \in (t_{j-1}, t_j]$, we have the following inequalities

$$\begin{aligned}
0 &\le \lambda_m \int_0^t \int_\Omega (\nabla \times \bar{\boldsymbol{B}}^n - \nabla \times \boldsymbol{B})^2 dxds \\
&\le \int_0^t \int_\Omega \bar{\nu}^n(t-\tau)(\nabla \times \bar{\boldsymbol{B}}^n - \nabla \times \boldsymbol{B})^2 dxds \\
&= \int_0^t (\bar{\nu}^n(t-\tau)\nabla \times \bar{\boldsymbol{B}}^n, \nabla \times \bar{\boldsymbol{B}}^n)ds + \int_0^t (\bar{\nu}^n(t-\tau)\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{B})ds \\
&- 2\int_0^t (\bar{\nu}^n(t-\tau)\nabla \times \bar{\boldsymbol{B}}^n, \nabla \times \boldsymbol{B})ds \\
&= \sum_{i=1}^3 \Pi_i.
\end{aligned}$$

From (22), we have

$$\begin{aligned}
\Pi_1 &= \int_0^t R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \bar{\boldsymbol{B}}^n \right) ds + \int_0^t \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \bar{\boldsymbol{B}}^n \right) ds \\
&- \int_0^t \left(\partial_t \boldsymbol{B}^n, \bar{\boldsymbol{B}}^n \right) ds \\
&= -\int_0^{t_j} \left(\partial_t \boldsymbol{B}^n, \bar{\boldsymbol{B}}^n \right) ds + \int_t^{t_j} (\partial_t \boldsymbol{B}^n, \bar{\boldsymbol{B}}^n) ds \\
&+ \int_0^t R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \bar{\boldsymbol{B}}^n \right) ds + \int_0^t \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \bar{\boldsymbol{B}}^n \right) ds \\
&= -\sum_{k=1}^j \int_\Omega \left(\boldsymbol{B}^k - \boldsymbol{B}^{k-1} \right) \boldsymbol{B}^k dx + \int_t^{t_j} \left(\partial_t \boldsymbol{B}^n, \bar{\boldsymbol{B}}^n \right) ds \\
&+ \int_0^t R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \bar{\boldsymbol{B}}^n \right) ds + \int_0^t \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \bar{\boldsymbol{B}}^n \right) ds \\
&\le -\int_\Omega \frac{{\boldsymbol{B}^j}^2 - {\boldsymbol{B}_0}^2}{2} dx + \int_t^{t_j} \left(\partial_t \boldsymbol{B}^n, \bar{\boldsymbol{B}}^n \right) ds + \int_0^t R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \bar{\boldsymbol{B}}^n \right) ds \\
&+ \int_0^t \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \bar{\boldsymbol{B}}^n \right) ds
\end{aligned}$$

$$= -\int_\Omega \frac{\bar{\boldsymbol{B}}^n(t)^2 - \boldsymbol{B}_0^2}{2} dx + \int_t^{t_j} (\partial_t \boldsymbol{B}^n, \bar{\boldsymbol{B}}^n) ds + \int_0^t R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \bar{\boldsymbol{B}}^n \right) ds$$
$$+ \int_0^t \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \bar{\boldsymbol{B}}^n \right) ds.$$

According to (26), (30), (31), (33), and (34), we have

$$\begin{aligned}
\lim_{n\to\infty} \Pi_1 \le & -\int_\Omega \frac{\boldsymbol{B}(t)^2 - \boldsymbol{B}_0^2}{2} dx + \int_0^t R_\alpha \Big(\frac{f\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}, \nabla \times \boldsymbol{B} \Big) ds \\
& + \int_0^t (\boldsymbol{U} \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) \, ds \\
& = -\int_0^t \int_\Omega \frac{1}{2} \frac{d\boldsymbol{B}^2}{ds} dx ds + \int_0^t R_\alpha \Big(\frac{f\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}, \nabla \times \boldsymbol{B} \Big) ds \\
& + \int_0^t (\boldsymbol{U} \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) ds \\
& = -\int_0^t (\partial_s \boldsymbol{B}, \boldsymbol{B}) ds + \int_0^t R_\alpha \Big(\frac{f\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}, \nabla \times \boldsymbol{B} \Big) ds \\
& + \int_0^t (\boldsymbol{U} \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) \, ds \\
& = \int_0^t (\nu(\xi) \nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) \, ds.
\end{aligned}$$

Based on (26) and (31), we get

$$\lim_{n\to\infty} \Pi_2 = \int_0^t (\bar{\nu}^n(t-\tau) \nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) \, ds = \int_0^t (\nu(\xi) \nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) \, ds,$$
$$\lim_{n\to\infty} \Pi_3 = -2 \int_0^t \left(\bar{\nu}^n (t-\tau) \nabla \times \bar{\boldsymbol{B}}^n, \nabla \times \boldsymbol{B} \right) ds$$
$$= -2 \int_0^t (\nu(\xi) \nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) \, ds.$$

Therefore, we have

$$0 \le \lambda_m \int_0^t \int_\Omega (\nabla \times \bar{\boldsymbol{B}}^n - \nabla \times \boldsymbol{B})^2 dx ds \le 0.$$

The above inequality is valid for any $t \in [0,T]$. Therefore we have

$$\nabla \times \bar{\boldsymbol{B}}^n \to \nabla \times \boldsymbol{B} \ \text{ in } \ L^2((0,T); L^2(\Omega)). \tag{35}$$

(Ⅲ) Let $\boldsymbol{\Phi} \in C_0^\infty(\bar{\Omega})$ in (22), then integrating it in $[0,\vartheta)$ where $\vartheta \in [0,T]$ yields

$$\begin{aligned}
& \int_0^\vartheta (\partial_t \boldsymbol{B}^n, \boldsymbol{\Phi}) \, \mathrm{d}t + \int_0^\vartheta \left(\bar{\nu}^n(t-\tau) \nabla \times \bar{\boldsymbol{B}}^n, \nabla \times \boldsymbol{\Phi} \right) \mathrm{d}t \\
= \ & \int_0^\vartheta R_\alpha \left(\frac{\bar{f}^n \bar{\boldsymbol{B}}^n}{1+\gamma|\bar{\boldsymbol{B}}^n|^2}, \nabla \times \boldsymbol{\Phi} \right) \mathrm{d}t + \int_0^\vartheta \left(\bar{\boldsymbol{U}}^n \times \bar{\boldsymbol{B}}^n, \nabla \times \boldsymbol{\Phi} \right) \mathrm{d}t.
\end{aligned}$$

Using (26), (31), (32), (33), and (34), we can obtain the limit for $n \to \infty$

$$\begin{aligned}
\int_0^\vartheta (\partial_t \boldsymbol{B}, \boldsymbol{\Phi}) \mathrm{d}t + \int_0^\vartheta (\nu \nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{\Phi}) \mathrm{d}t \ = \ & \int_0^\vartheta R_\alpha \Big(\frac{f\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}, \nabla \times \boldsymbol{\Phi} \Big) \mathrm{d}t \\
+ \ & \int_0^\vartheta (\boldsymbol{U} \times \boldsymbol{B}, \nabla \times \boldsymbol{\Phi}) \mathrm{d}t.
\end{aligned}$$

Using the fact that $C_0^\infty(\bar{\Omega})$ is dense in $H_0(\mathrm{curl};\Omega)$ and differentiating with respect to time variable, we know that $\boldsymbol{B}$ and ξ satisfy (11).

Now, we integrate (23) in time

$$(\bar{\xi}^n(s),\Upsilon)-(\xi^n(0),\Upsilon)+(\xi^n(s)-\bar{\xi}^n(s),\Upsilon)+\int_0^s(\kappa\nabla\bar{\xi}^n,\nabla\Upsilon)\mathrm{d}t$$
$$=\int_0^s(\mathcal{C}_r(\bar{q}^n(t-\tau)K(\bar{\boldsymbol{B}}^n)),\Upsilon)\mathrm{d}t-\int_0^s(\kappa\nabla\theta_0,\nabla\Upsilon)\mathrm{d}t. \tag{36}$$

Due to $|(\xi^n(s)-\bar{\xi}^n(s),\Upsilon)|\le\tau|(\partial_t\xi^n,\Upsilon)|\le\tau\|\partial_t\xi^n\|\|\Upsilon\|$ and Lemma 3.3, we have

$$\lim_{n\to\infty}(\xi^n(s)-\bar{\xi}^n(s),\Upsilon)=0,\quad\text{for any } s\in[0,T].$$

Based on (24), (25), (30), and (35), and the limit for $n\to\infty$ in (36), we obtain

$$(\xi(s),\Upsilon)-(\xi(0),\Upsilon)+\int_0^s(\kappa\nabla\xi,\nabla\Upsilon)\mathrm{d}t=\int_0^s(\mathcal{C}_r(q(\xi)K(\boldsymbol{B})),\Upsilon)\mathrm{d}t-\int_0^s(\kappa\nabla\theta_0,\nabla\Upsilon)\mathrm{d}t.$$

Then differentiating in time, $\boldsymbol{B}$ and ξ solve (12).

■

4. Convergence analysis of the full-discrete scheme finite element methods

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of cube in 3D. For every element $K\in\mathcal{T}_h$, h_K denotes the diameter of a generic element $K\in\mathcal{T}_h$, $h=\max_{K\in\mathcal{T}_h}h_K$ denotes the mesh size. Now, the Nédélec's element space $\boldsymbol{V}_h$ [22] and Lagrange finite element space W_h are shown as follows

$$\mathbf{V}_h=\{\boldsymbol{v}_h\in H(\mathrm{curl};\Omega):\ \boldsymbol{v}_h|_K\in Q_{p-1,p,p}\times Q_{p,p-1,p}\times Q_{p,p,p-1},\ \forall K\in\mathcal{T}_h\},$$
$$\mathbf{V}_h^0=\{\boldsymbol{v}_h\in\mathbf{V}_h,\ \boldsymbol{n}\times\boldsymbol{v}_h=0\text{ on }\partial\Omega\},$$
$$W_h=\left\{w_h\in H^1(\Omega):\ w_h|_K\in Q_{p,p,p}\right\},$$
$$W_h^0=\{w_h\in W_h,w_h|_{\Gamma_1}=0\}.$$

Here and hereafter $Q_{i,j,m}$ means the space of polynomials whose degrees are less than or equal to i,j,m in variables x,y,z, respectively. Hence, the full-discrete variational formulations can be simulated: Find $\boldsymbol{B}_h^k\in\boldsymbol{V}_h^0$, $\xi_h^k\in W_h^0$ such that

$$(\delta_\tau\boldsymbol{B}_h^k,\boldsymbol{\Phi}_h)+(\nu(\xi_h^{k-1})\nabla\times\boldsymbol{B}_h^k,\nabla\times\boldsymbol{\Phi}_h)=R_\alpha\left(\frac{f^k\boldsymbol{B}_h^k}{1+\gamma|\boldsymbol{B}_h^{k-1}|^2},\nabla\times\boldsymbol{\Phi}_h\right)$$
$$+(\boldsymbol{U}^k\times\boldsymbol{B}_h^k,\nabla\times\boldsymbol{\Phi}_h),\quad\forall\,\boldsymbol{\Phi}_h\in\boldsymbol{V}_h^0, \tag{37}$$

$$(\delta_\tau\xi_h^k,\Upsilon_h)+(\kappa\nabla\xi_h^k,\nabla\Upsilon_h)=\left(\mathcal{C}_r\left(q(\xi_h^{k-1})K\left(\boldsymbol{B}_h^k\right)\right),\Upsilon_h\right)-(\kappa\nabla\theta_0,\nabla\Upsilon_h),\ \forall\,\Upsilon_h\in W_h^0, \tag{38}$$

with the initial conditions

$$\boldsymbol{B}_h^0(\boldsymbol{x})=\Pi_c\boldsymbol{B}_0(\boldsymbol{x}),\quad\xi_h^0(\boldsymbol{x})=\Pi_h\xi_0(\boldsymbol{x}), \tag{39}$$

where $f^k=f(\boldsymbol{x},k\tau)$, $\boldsymbol{U}^k=\boldsymbol{U}(\boldsymbol{x},k\tau)$, Π_c is the so-called Nédélec interpolation operator [14], and Π_h is the standard Lagrange interpolation operator.

The existence and uniqueness of $(\boldsymbol{B}_h^k,\xi_h^k)$ in (37)-(38) is obvious since it has became a linear decoupled problem.

Similar to the estimates (17) and (19), we have the next lemma.

Lemma 4.1. Assume that $\left(\boldsymbol{B}_h^k, \xi_h^k\right)$ is the solution of the discrete system (37)-(38) for each fixed k $(1 \le k \le n)$, then the sequences $\{\boldsymbol{B}_h^k\}_{k=1}^n$ and $\{\xi_h^k\}_{k=1}^n$ have the following stability estimates

$$\max_{1\le k\le n} \left\|\boldsymbol{B}_h^k\right\|^2 + \sum_{k=1}^n \tau\lambda_m \left\|\nabla\times \boldsymbol{B}_h^k\right\|^2 \le C\left\|\boldsymbol{B}_h^0\right\|^2, \tag{40}$$

$$\max_{1\le k\le n} \left\|\xi_h^k\right\|^2 + \sum_{k=1}^n \tau\kappa \left\|\nabla\xi_h^k\right\|^2 \le C(\left\|\xi_h^0\right\|^2 + \|\nabla\theta_0\|^2). \tag{41}$$

Now, we only consider the part that the cut-off function satisfies $\mathcal{C}_r(\boldsymbol{x}) = \boldsymbol{x}$. First, we state the second main result about the error estimate after combining the following Lemma 4.2 and Lemma 4.3.

Theorem 4.1. *Let $(\boldsymbol{B}^k, \xi^k)$ and $(\boldsymbol{B}_h^k, \xi_h^k)$ be the solutions of (11)-(12) and (37)-(38), respectively. Assume that $\boldsymbol{B}_t,\ \nabla\times\boldsymbol{B}_t,\ \boldsymbol{B}, \nabla\times\boldsymbol{B} \in L^\infty(0,T;\boldsymbol{H}^s(\Omega))$, $\xi,\ \xi_t \in L^\infty(0,T;H^{m+1}(\Omega))$, where $1\le m\le p$, $\frac{1}{2}+\delta \le s\le p$, $0<\delta<\frac{1}{2}$, we have*

$$\begin{aligned}&\max_{1\le k\le n} \left\|\boldsymbol{B}^k - \boldsymbol{B}_h^k\right\|^2_{\boldsymbol{L}^2(\Omega)} + \max_{1\le k\le n}\left\|\xi^k-\xi_h^k\right\|^2_{L^2(\Omega))} + \lambda_m \left\|\nabla\times(\boldsymbol{B}^k-\boldsymbol{B}_h^k)\right\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\ &\qquad + \kappa\left\|\nabla\xi^k - \nabla\xi_h^k\right\|^2_{L^2(0,T;L^2(\Omega))} \le C\left(h^{2\min\{s,m\}}+\tau^2\right)\Lambda\left(\boldsymbol{B},\xi\right),\end{aligned} \tag{42}$$

where C is a positive constant independent of the mesh size h and time step τ. And $\Lambda(\boldsymbol{B},\xi)$ is defined by (45).

Firstly, we give the interpolation theorem on the space $\boldsymbol{V}_h$ and the interpolation results for ξ can be find in [4, chap 4].

Lemma 4.2. ([18]) Assume that $0<\delta<\frac{1}{2}$ and $\mathcal{T}_h$ is a regular family of hexahedral meshes on Ω with faces aligning with the coordinate axes. If $\boldsymbol{B}, \nabla\times\boldsymbol{B}\in\boldsymbol{H}^s(\Omega)$, $\frac{1}{2}+\delta\le s\le p$, then there exsits a constant $C>0$ independent of h and $\boldsymbol{B}$ such that

$$\|\boldsymbol{B}-\Pi_c\boldsymbol{B}\|_0 + \|\nabla\times(\boldsymbol{B}-\Pi_c\boldsymbol{B})\|_0 \le Ch^s\left(\|\boldsymbol{B}\|_{\boldsymbol{H}^s(\Omega)} + \|\nabla\times\boldsymbol{B}\|_{\boldsymbol{H}^s(\Omega)}\right). \tag{43}$$

Secondly, we prove the approximation properties between the interpolations and finite element solutions.

Lemma 4.3. Under the assumption of Theorem 4.1, we have

$$\begin{aligned}&\max_{1\le k\le n}\left\|\boldsymbol{\eta}_h^k\right\|^2_{\boldsymbol{L}^2(\Omega)} + \max_{1\le k\le n}\left\|\zeta_h^k\right\|^2_{L^2(\Omega))} + \lambda_m\sum_{k=1}^m \tau\left\|\nabla\times\boldsymbol{\eta}_h^k\right\|^2_{\boldsymbol{L}^2(\Omega)}\\ &+\kappa\sum_{k=1}^m\tau\left\|\nabla\zeta_h^k\right\|^2_{L^2(\Omega)} \le C\left(h^{2\min\{s,m\}}+\tau^2\right)\Lambda\left(\boldsymbol{B},\xi\right),\end{aligned} \tag{44}$$

where $\boldsymbol{\eta}_h^k = \boldsymbol{B}_h^k - \Pi_c\boldsymbol{B}^k$, $\zeta_h^k = \xi_h^k - \Pi_h\xi^k$, and

$$\begin{aligned}\Lambda(\boldsymbol{B},\xi) = &\|\boldsymbol{B}_t\|^2_{L^\infty(0,T;\boldsymbol{H}^s(\Omega))} + \|\nabla\times\boldsymbol{B}_t\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{H}^s(\Omega))}\\ &+\|\nabla\times\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{H}^s(\Omega))} + (\|\xi_t\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\xi\|^2_{L^\infty(0,T;H^{m+1}(\Omega))})\end{aligned}$$

$$
\begin{aligned}
&\cdot \|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\xi_t\|^2_{L^\infty(0,T;H^{m+1}(\Omega))} + \|\xi\|^2_{L^\infty(0,T;H^{m+1}(\Omega))}\\
&+ \|\xi_t\|^2_{L^\infty(0,T;L^2(\Omega))}\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}(\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\
&+ \|\xi\|^2_{L^\infty(0,T;L^2(\Omega))}\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}(\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\
&+ \|\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}) + \|\nabla \times \boldsymbol{B}_t\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}(\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\
&+ \|\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}) + \|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}(\|\boldsymbol{B}_t\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\
&+ \|\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}).
\end{aligned}
\tag{45}
$$

Proof. Let $\boldsymbol{\Phi} = \boldsymbol{\eta}_h^k \in V_h^0$ in (11), then integrating over $[t_{k-1}, t_k]$ and subtracting it from (37), it yields

$$
\begin{aligned}
&\tau\left(\delta_\tau \boldsymbol{\eta}_h^k, \boldsymbol{\eta}_h^k\right) + \tau\left(\nu\left(\xi_h^{k-1}\right)\nabla \times \boldsymbol{\eta}_h^k, \nabla \times \boldsymbol{\eta}_h^k\right)\\
&= \tau\left(\delta_\tau\left(\boldsymbol{B}^k - \Pi_c \boldsymbol{B}^k\right), \boldsymbol{\eta}_h^k\right) + \tau\bigg(\frac{1}{\tau}\int_{t_{k-1}}^{t_k} \nu(\xi)\nabla \times \boldsymbol{B}\mathrm{d}t - \nu(\xi_h^{k-1})\nabla \times \Pi_c \boldsymbol{B}^k,\\
&\nabla \times \boldsymbol{\eta}_h^k\bigg) + R_\alpha \tau\bigg(\frac{f(\boldsymbol{x}, k\tau)\boldsymbol{B}_h^k}{1+\gamma|{\boldsymbol{B}_h}^{k-1}|^2} - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{f(\boldsymbol{x},t)\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2}\mathrm{d}t, \nabla \times \boldsymbol{\eta}_h^k\bigg)\\
&+ \tau\bigg(\boldsymbol{U}^k \times \boldsymbol{B}_h^k - \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{U} \times \boldsymbol{B}\mathrm{d}t, \nabla \times \boldsymbol{\eta}_h^k\bigg)\\
&= \sum_{i=1}^4 Err_i.
\end{aligned}
\tag{46}
$$

Based on interpolation theorem, we have

$$
Err_1 = \tau(\delta_\tau(\boldsymbol{B}^k - \Pi_c \boldsymbol{B}^k), \boldsymbol{\eta}_h^k) \le C\tau h^s \|\boldsymbol{B}_t\|_{L^\infty(0,T;\boldsymbol{H}^s(\Omega))}\|\boldsymbol{\eta}_h^k\|_0. \tag{47}
$$

Thanks to the Lipschitz continuity of ν and Taylor expansion, we arrive at

$$
\begin{aligned}
Err_2 \;=\;& \bigg(\int_{t_{k-1}}^{t_k} (\nu(\xi) - \nu(\xi_h^{k-1}))\nabla \times \boldsymbol{B}\mathrm{d}t, \nabla \times \boldsymbol{\eta}_h^k\bigg)\\
+\;& \bigg(\int_{t_{k-1}}^{t_k} \nu(\xi_h^{k-1})(\nabla \times \boldsymbol{B} - \nabla \times \Pi_c \boldsymbol{B}^k)\mathrm{d}t, \nabla \times \boldsymbol{\eta}_h^k\bigg)\\
\le\;& C\left(\tau^2\|\xi_t\|_{L^\infty(0,T;L^2(\Omega))} + \tau h^{m+1}\|\xi\|_{L^\infty(0,T;H^{m+1}(\Omega))} + \tau\|\zeta_h^{k-1}\|_0\right)\\
\cdot\|\;& \nabla \times \boldsymbol{B}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla \times \boldsymbol{\eta}_h^k\|_0 + C\lambda_M(\tau^2\|\nabla \times \boldsymbol{B}_t\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\
+\;& \tau h^s(\|\boldsymbol{B}\|_{L^\infty(0,T;\boldsymbol{H}^s(\Omega))} + \|\nabla \times \boldsymbol{B}\|_{L^\infty(0,T;\boldsymbol{H}^s(\Omega))}))\|\nabla \times \boldsymbol{\eta}_h^k\|_0.
\end{aligned}
\tag{48}
$$

Furthermore, there holds

$$
\begin{aligned}
Err_3 \;=\;& R_\alpha\bigg(\int_{t_{k-1}}^{t_k} \frac{f^k}{1+\gamma|\boldsymbol{B}_h^{k-1}|^2}(\boldsymbol{B}_h^k - \Pi_c \boldsymbol{B}^k + \Pi_c \boldsymbol{B}^k - \boldsymbol{B})\mathrm{d}t\\
+\;& \int_{t_{k-1}}^{t_k} \frac{(1+\gamma|\boldsymbol{B}|^2)(f^k - f) + \gamma f(|\boldsymbol{B}|^2 - |\boldsymbol{B}_h^{k-1}|^2)}{(1+\gamma|\boldsymbol{B}_h^{k-1}|^2)(1+\gamma|\boldsymbol{B}|^2)}\boldsymbol{B}\mathrm{d}t, \nabla \times \boldsymbol{\eta}_h^k\bigg).
\end{aligned}
\tag{49}
$$

In order to obtain the estimation of Err_3, we have

$$\begin{aligned}
&\left|\int_{t_{k-1}}^{t_k} \frac{(1+\gamma|\boldsymbol{B}|^2)(f^k-f)}{(1+\gamma|\boldsymbol{B}_h^{k-1}|^2)(1+\gamma|\boldsymbol{B}|^2)}\boldsymbol{B}\mathrm{d}t\right| \\
\leq\ & \int_{t_{k-1}}^{t_k} |f^k-f||\boldsymbol{B}|\mathrm{d}t \leq \left(\int_{t_{k-1}}^{t_k}|f^k-f|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k}|\boldsymbol{B}|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
\leq\ & f_M\tau^2\|\boldsymbol{B}\|_{L^\infty(0,T)}, \qquad (50)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{t_{k-1}}^{t_k} \frac{\gamma f(|\boldsymbol{B}|^2-|\boldsymbol{B}_h^{k-1}|^2)}{(1+\gamma|\boldsymbol{B}_h^{k-1}|^2)(1+\gamma|\boldsymbol{B}|^2)}\boldsymbol{B}\mathrm{d}t\right| \\
=\ & \left|\int_{t_{k-1}}^{t_k} \frac{\gamma f(|\boldsymbol{B}|-|\boldsymbol{B}_h^{k-1}|)(|\boldsymbol{B}|+|\boldsymbol{B}_h^{k-1}|)}{(1+\gamma|\boldsymbol{B}_h^{k-1}|^2)(1+\gamma|\boldsymbol{B}|^2)}\boldsymbol{B}\mathrm{d}t\right| \\
\leq\ & 2\int_{t_{k-1}}^{t_k}|f|\,|\boldsymbol{B}-\boldsymbol{B}^{k-1}+\boldsymbol{B}^{k-1}-\Pi_c\boldsymbol{B}^{k-1}+\Pi_c\boldsymbol{B}^{k-1}-\boldsymbol{B}_h^{k-1}|\mathrm{d}t \\
\leq\ & 2f_M\left(\tau^2\|\boldsymbol{B}_t(t)\|_{L^\infty(0,T)}+\tau|\boldsymbol{B}^{k-1}-\Pi_c\boldsymbol{B}^{k-1}|+\tau|\boldsymbol{\eta}_h^{k-1}|\right), \qquad (51)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k}\frac{f^k}{1+\gamma|\boldsymbol{B}_h^{k-1}|^2}(\boldsymbol{B}_h^k-\Pi_c\boldsymbol{B}^k+\Pi_c\boldsymbol{B}^k-\boldsymbol{B})\mathrm{d}t \\
\leq\ & f_M\left(\tau|\boldsymbol{\eta}_h^k|+\tau|\Pi_c\boldsymbol{B}^k-\boldsymbol{B}^k|+\tau^2\|\boldsymbol{B}_t(t)\|_{L^\infty(0,T)}\right). \qquad (52)
\end{aligned}$$

Then, substituting (50)-(52) into (49), we have

$$\begin{aligned}
Err_3 \leq\ & 2f_MR_\alpha(\tau\|\Pi_c\boldsymbol{B}^k-\boldsymbol{B}^k\|_0+\tau\|\Pi_c\boldsymbol{B}^{k-1}-\boldsymbol{B}^{k-1}\|_0+\tau^2\|\boldsymbol{B}_t\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
+\ & \tau^2\|\boldsymbol{B}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))})\|\nabla\times\boldsymbol{\eta}_h^k\|_0+\frac{2\tau f_M^2R_\alpha^2}{\lambda_m}\|\boldsymbol{\eta}_h^k\|_0^2+\frac{\tau\lambda_m}{8}\|\nabla\times\boldsymbol{\eta}_h^k\|_0^2 \\
+\ & \frac{4\tau f_M^2R_\alpha^2}{\lambda_m}\|\boldsymbol{\eta}_h^{k-1}\|_0^2+\frac{\tau\lambda_m}{8}\|\nabla\times\boldsymbol{\eta}_h^k\|_0^2. \qquad (53)
\end{aligned}$$

As for Err_4, we know that

$$Err_4=\left(\int_{t_{k-1}}^{t_k}\boldsymbol{U}^k\times\boldsymbol{B}_h^k-\boldsymbol{U}^k\times\boldsymbol{B}+\boldsymbol{U}^k\times\boldsymbol{B}-\boldsymbol{U}\times\boldsymbol{B}\mathrm{d}t,\nabla\times\boldsymbol{\eta}_h^k\right). \qquad (54)$$

Divided (54) into two parts, we get

$$\begin{aligned}
&\left|\int_{t_{k-1}}^{t_k}\boldsymbol{U}^k\times\boldsymbol{B}_h^k-\boldsymbol{U}^k\times\boldsymbol{B}\mathrm{d}t\right| \\
\leq\ & u_M\tau\left(|\boldsymbol{\eta}_h^k|+|\Pi_c\boldsymbol{B}^k-\boldsymbol{B}^k|+\tau\|\boldsymbol{B}_t(t)\|_{L^\infty(0,T)}\right), \qquad (55)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{t_{k-1}}^{t_k}\boldsymbol{U}^k\times\boldsymbol{B}-\boldsymbol{U}\times\boldsymbol{B}\mathrm{d}t\right| \\
\leq\ & \int_{t_{k-1}}^{t_k}|\boldsymbol{U}^k-\boldsymbol{U}||\boldsymbol{B}|\mathrm{d}t \\
\leq\ & \left(\int_{t_{k-1}}^{t_k}\tau^2|\partial_t\boldsymbol{U}^k|^2\mathrm{d}t\right)^{\frac{1}{2}}\sqrt{\tau}\|\boldsymbol{B}(t)\|_{L^\infty(0,T)}. \qquad (56)
\end{aligned}$$

Combining (55) with (56), we have

$$\begin{aligned} Err4 &\le u_M\left(\tau^2\|\boldsymbol{B}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\tau^2\|\boldsymbol{B}_t\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\right. \\ &+ \left.\tau\|\Pi_c\boldsymbol{B}^k-\boldsymbol{B}^k\|_0\right)\|\nabla\times\boldsymbol{\eta}_h^k\|_0+\frac{2u_M^2\tau}{\lambda_m}\|\boldsymbol{\eta}_h^k\|_0^2+\frac{\tau\lambda_m}{8}\|\nabla\times\boldsymbol{\eta}_h^k\|_0^2. \end{aligned}$$

Let $\Upsilon=\zeta_h^k\in W_h^0$ in (12), then integrating over $[t_{k-1},t_k]$ and subtracting it from (38), we obtain

$$\begin{aligned} &\tau\left(\delta_\tau\zeta_h^k,\zeta_h^k\right)+\tau\left(\kappa\nabla\zeta_h^k,\nabla\zeta_h^k\right) \\ &=\tau\left(\delta_\tau(\xi^k-\Pi\xi^k),\zeta_h^k\right)+\tau\left(\kappa\nabla\left(\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\xi\mathrm{d}t-\Pi\xi^k\right),\nabla\zeta_h^k\right) \\ &+\tau\left(\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\mathcal{C}_r(q(\xi_h^{k-1})K(\boldsymbol{B}_h^k))-q(\xi)K(\boldsymbol{B})\mathrm{d}t,\zeta_h^k\right) \\ &=\sum_{i=5}^{7}Err_i. \end{aligned} \tag{57}$$

In order to ensure the boundedness of ξ in L^∞-norm, we need the following assumption.

A priori L^∞ assumption up to time step $t_i,\ i\le k-1$. Assume that an L^∞ bound for the exact solution and its interpolation satisfies

$$\left\|\xi^i\right\|_{L^\infty(\Omega)}\le C^*,\quad \left\|\Pi_h\xi^i\right\|_{L^\infty(\Omega)}\le C^*, \tag{58}$$

where C^* is a positive constant. Note that the second inequality comes from the following estimate

$$\left\|\xi^i-\Pi_h\xi^i\right\|_{L^\infty(\Omega)}\le Ch^{m+1}|\ln h|. \tag{59}$$

We also assume that the numerical error function for ξ has an L^∞ bound at time step t_i

$$\left\|e^i\right\|_{L^\infty(\Omega)}:=\left\|\Pi_h\xi^i-\xi_h^i\right\|_{L^\infty(\Omega)}\le 1, \tag{60}$$

so that an L^∞ bound for the numerical solution ξ_h^i is available, i.e.,

$$\left\|\xi_h^i\right\|_{L^\infty(\Omega)}=\left\|\Pi_h\xi^i-e^i\right\|_{L^\infty(\Omega)}=\left\|\Pi_h\xi^i\right\|_{L^\infty(\Omega)}+\left\|e^i\right\|_{L^\infty(\Omega)}\le\tilde{C}_0, \tag{61}$$

where $\tilde{C}_0=C^*+1$. This assumption will be recovered in later analysis.

Similar to the derivations of (47) and (48), we get

$$Err_5=\tau\left(\delta_\tau(\xi^k-\Pi\xi^k),\zeta_h^k\right)\le C\tau h^{m+1}\left\|\xi_t\right\|_{L^\infty(0,T;H^{m+1}(\Omega))}\left\|\zeta_h^k\right\|_0, \tag{62}$$

and

$$\begin{aligned} Err_6 &= \tau\left(\kappa\nabla\left(\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\xi\mathrm{d}t-\xi^k\right),\nabla\zeta_h^k\right)+\tau\left(\kappa\nabla(\xi^k-\Pi\xi^k),\nabla\zeta_h^k\right) \\ &\le \kappa\left(\tau^2\|\xi_t\|_{L^\infty(0,T;H^1(\Omega))}+\tau h^m\|\xi\|_{L^\infty(0,T;H^{m+1}(\Omega))}\right)\|\nabla\zeta_h^k\|_0. \end{aligned} \tag{63}$$

For Err_7, we have

$$\begin{aligned} Err_7=&\left(\int_{t_{k-1}}^{t_k}\left(q\left(\xi\right)-q\left(\xi_h^{k-1}\right)\right)K(\boldsymbol{B})\mathrm{d}t\right. \\ &\left.+\int_{t_{k-1}}^{t_k}q\left(\xi_h^{k-1}\right)\left(K\left(\boldsymbol{B}\right)-K\left(\boldsymbol{B}_h^k\right)\right)\mathrm{d}t,\zeta_h^k\right) \end{aligned}$$

$$:= (I_1 + I_2, \zeta_h^k).$$

Meanwhile, the estimates for I_1 and I_2 are deduced as follows

$$\begin{aligned}
I_1 \quad &\leq \quad \left| M \int_{t_{k-1}}^{t_k} (\xi_h^{k-1} - \xi) K(\boldsymbol{B}) \mathrm{d}t \right| \\
&\leq \quad C\left(\tau \|\zeta_h^{k-1}\| + \tau |\Pi \xi^{k-1} - \xi^{k-1}| + \tau^2 \|\xi_t\|_{L^\infty(0,T)}\right) \Big(\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T)} \\
&+ \quad \|\nabla \times \boldsymbol{B}\|_{L^\infty(0,T)} \|\boldsymbol{B}\|_{L^\infty(0,T)} \Big),
\end{aligned}$$

and

$$\begin{aligned}
I_2 \quad &= \quad \left| \int_{t_{k-1}}^{t_k} q(\xi_h^{k-1})(K(\boldsymbol{B}) - K(\boldsymbol{B}^k) + K(\boldsymbol{B}^k) - K(\boldsymbol{B}_h^k)) \mathrm{d}t \right| \\
&\leq \quad \sigma_M \tau \left| K(\boldsymbol{B}_h^k) - K(\boldsymbol{B}^k) \right| + \sigma_M \left| \int_{t_{k-1}}^{t_k} K(\boldsymbol{B}^k) - K(\boldsymbol{B}) \mathrm{d}t \right|.
\end{aligned}$$

To estimate I_2, we have

$$\begin{aligned}
&\left| K(\boldsymbol{B}_h^k) - K(\boldsymbol{B}^k) \right| \leq \left| \nabla \times \boldsymbol{B}_h^k - \nabla \times \boldsymbol{B}^k \right| \left(\left| \nabla \times \boldsymbol{B}_h^k \right| + \left| \nabla \times \boldsymbol{B}^k \right| \right) + \big| \boldsymbol{U}^k \times \boldsymbol{B}^k \nabla \times \boldsymbol{B}^k \\
&\qquad - \boldsymbol{U}^k \times \boldsymbol{B}^k \nabla \times \boldsymbol{B}_h^k + \boldsymbol{U}^k \times \boldsymbol{B}^k \nabla \times \boldsymbol{B}_h^k - \boldsymbol{U}^k \times \boldsymbol{B}_h^k \nabla \times \boldsymbol{B}_h^k \big| \\
&\qquad + R_\alpha \bigg| \nabla \times \boldsymbol{B}^k \Big(\frac{f(x,k\tau)\boldsymbol{B}^k}{1+\gamma|\boldsymbol{B}^k|^2} \Big) - \nabla \times \boldsymbol{B}_h^k \Big(\frac{f(x,k\tau)\boldsymbol{B}^k}{1+\gamma|\boldsymbol{B}^k|^2} \Big) \\
&\qquad + \nabla \times \boldsymbol{B}_h^k \Big(\frac{f(x,k\tau)\boldsymbol{B}^k}{1+\gamma|\boldsymbol{B}^k|^2} \Big) - \nabla \times \boldsymbol{B}_h^k \Big(\frac{f(x,k\tau)\boldsymbol{B}_h^k}{1+\gamma|\boldsymbol{B}_h^k|^2} \Big) \bigg| \\
&\qquad \leq C\Big(\left| \nabla \times \left(\boldsymbol{B}_h^k - \boldsymbol{B}^k \right) \right| + (u_M + R_\alpha f_M) \left| \nabla \times \left(\boldsymbol{B}_h^k - \boldsymbol{B}^k \right) \right| \\
&\qquad + (u_M + R_\alpha f_M) \left| \boldsymbol{B}_h^k - \boldsymbol{B}^k \right| \Big),
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_M \left| \int_{t_{k-1}}^{t_k} K(\boldsymbol{B}^k) - K(\boldsymbol{B}) \mathrm{d}t \right| \\
&= \sigma_M \bigg| \int_{t_{k-1}}^{t_k} (|\nabla \times \boldsymbol{B}^k|^2 - |\nabla \times \boldsymbol{B}|^2) - (\nabla \times \boldsymbol{B}^k \cdot (\boldsymbol{U} \times \boldsymbol{B}^k) - \nabla \times \boldsymbol{B} \cdot (\boldsymbol{U} \times \boldsymbol{B})) \\
&- \bigg(R_\alpha \nabla \times \boldsymbol{B}^k \cdot \Big(\frac{f(\boldsymbol{x},k\tau)\boldsymbol{B}^k}{1+\gamma|\boldsymbol{B}^k|^2} \Big) - R_\alpha \nabla \times \boldsymbol{B} \cdot \Big(\frac{f(\boldsymbol{x},t)\boldsymbol{B}}{1+\gamma|\boldsymbol{B}|^2} \Big) \bigg) \mathrm{d}t \bigg| \\
&:= \sigma_M (i_1 + i_2 + i_3).
\end{aligned}$$

Here, we reach the following inequalities

$$\begin{aligned}
|i_1| \quad &\leq \quad \left| \int_{t_{k-1}}^{t_k} (\nabla \times \boldsymbol{B}^k - \nabla \times \boldsymbol{B}) \nabla \times \boldsymbol{B}^k + (\nabla \times \boldsymbol{B}^k - \nabla \times \boldsymbol{B}) \nabla \times \boldsymbol{B} \mathrm{d}t \right| \\
&\leq \quad \tau^2 \|\nabla \times \boldsymbol{B}_t\|_{L^\infty(0,T)} \|\nabla \times \boldsymbol{B}\|_{L^\infty(0,T)}, \\
|i_2| \quad &\leq \quad u_M \tau^2 \|\nabla \times \boldsymbol{B}_t\|_{L^\infty(0,T)} \|\boldsymbol{B}\|_{L^\infty(0,T)} + u_M \tau^2 \|\boldsymbol{B}_t\|_{L^\infty(0,T)} \|\nabla \times \boldsymbol{B}\|_{L^\infty(0,T)}, \\
|i_3| \quad &\leq \quad C\tau^2 \|\nabla \times \boldsymbol{B}_t\|_{L^\infty(0,T)} \|\boldsymbol{B}\|_{L^\infty(0,T)} + \tau^2 \|\nabla \times \boldsymbol{B}\|_{L^\infty(0,T)} \|\boldsymbol{B}_t\|_{L^\infty(0,T)} \\
&+ \quad C\tau^2 \|f_t\|_{L^\infty(0,T)} \|\boldsymbol{B}\|_{L^\infty(0,T)} \|\nabla \times \boldsymbol{B}\|_{L^\infty(0,T)}.
\end{aligned}$$

By Lemma 4.2, Young's inequality, summing both sides of (46) and (57) together over $k = 1, 2, \cdots, n$, using the fact $n\tau \le T$ and the estimates results $Err_i,\ i = 1, \cdots, 7$, choosing $0 < \tau < C \min\Big\{(4\|\nabla \times \boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))})^{-1}(T\sigma_M^2, 2T\sigma_M^2 u_M^2, 2T\sigma_M^2 R_\alpha^2 f_M^2,$ $\|\nabla\times\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{B}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))})^{-1}, \dfrac{\lambda_m}{3}(4f_M^2R_\alpha^2, 2u_M^2)^{-1}\Big\}$, and employing the discrete Grönwall inequality, we finish the estimates (44).

Recovery of the priori bound. Define Ψ to be the solution for the elliptic equation

$$-\triangle\Psi = \xi_h^k - \Pi\xi^k.$$

with Dirichlet boundary condition $\Psi|_{\partial\Omega} = 0$. Using the Aubin-Nitsche technique and (44), we can get

$$\left\|\xi_h^k - \Pi\xi^k\right\|_0 \le C(h^{\min\{m,s\}+1} + \tau). \tag{64}$$

With the help of the inverse inequality and an application of the L^2 error estimate (64), the following estimate is available, for $d \le 3$:

$$\left\|\xi_h^k - \Pi\xi^k\right\|_{L^\infty} \le \frac{C\left\|\xi_h^k - \Pi\xi^k\right\|_0}{h^{d/2}} \le \frac{C\left(\tau + h^{\min\{m,s\}+1}\right)}{h^{d/2}},$$

under the requirement $\tau = O(h^{\frac{d}{2}+\epsilon})$ for any $0 < \epsilon < 1$. Then we complete the recovery. ■

5. Numerical test

In this section, we main to verify our theoretical analysis about the convergence, i.e., Theorem 4.1. For simplicity, we assume that $\Omega = [0,1]^3$ and fix the time step $\tau = 10^{-5}$ on uniform mesh. Moreover, we choose $\lambda(\theta) = \frac{\theta^2}{1+\theta^2}$, $\sigma(\theta) = e^{-\theta}$, $\kappa = 1$, $f(\boldsymbol{x}, t) = 1$, $\boldsymbol{U} = [1,1,1]^T$. The analytical solutions of (1)-(2) are given as follows

$$\boldsymbol{B}(\boldsymbol{x}, t) = \begin{pmatrix} \boldsymbol{B}_x \\ \boldsymbol{B}_y \\ \boldsymbol{B}_z \end{pmatrix} = e^{-t}\cos t \begin{pmatrix} \cos\pi x \sin\pi y \sin\pi z \\ \frac{1}{3}\sin\pi x\cos\pi y\sin\pi z \\ -\frac{4}{3}\sin\pi x\sin\pi y\cos\pi z \end{pmatrix},$$

$$\theta(\boldsymbol{x}, t) = e^{t(x-1)x(y-1)y(z-1)z^2}.$$

Cubic meshes are used in this part contains $i \times j \times k$ elements, where (i, j, k) indicates the number of divisions in x, y, and z directions, respectively. The results in Table 1 show that the optimal rates of convergence are achieved, which are consistent with Theorem 4.1. We denote three errors by $Err_1 = \|\boldsymbol{B} - \boldsymbol{B}_h\|_0$, $Err_2 = \|\nabla \times (\boldsymbol{B} - \boldsymbol{B}_h)\|_0$, and $Err_3 = \|\theta - \theta_h\|_0$. Then we have following results.

TABLE 1. Convergence of $\boldsymbol{B}$ and θ after 100 time steps.

meshes	Err_1	rates	Err_2	rates	Err_3	rates
$10\times10\times10$	5.4445e-02	-	4.1842e-01	-	6.8549e-05	-
$15\times15\times15$	3.6296e-02	1.0001	2.7914e-01	0.9982	3.0812e-05	1.9722
$20\times20\times20$	2.7222e-02	1.0000	2.0941e-01	0.9991	1.7397e-05	1.9868
$25\times25\times25$	2.1777e-02	1.0000	1.6755e-01	0.9995	1.1157e-05	1.9909
$30\times30\times30$	1.8148e-02	1.0000	1.3963e-01	0.9997	7.7599e-06	1.9915
$35\times35\times35$	1.5556e-02	1.0000	1.1969e-01	0.9998	5.7093e-06	1.9907
$40\times40\times40$	1.3611e-02	1.0000	1.0473e-01	0.9998	4.3776e-06	1.9890

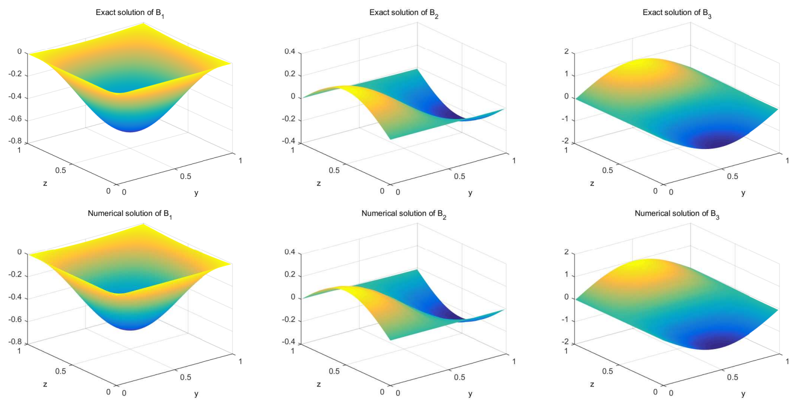

FIGURE 1. The true solution (up) and the numerical solution (down) of three components of $\boldsymbol{B}$ with $\tau = 10^{-5}$ after 100 time steps.

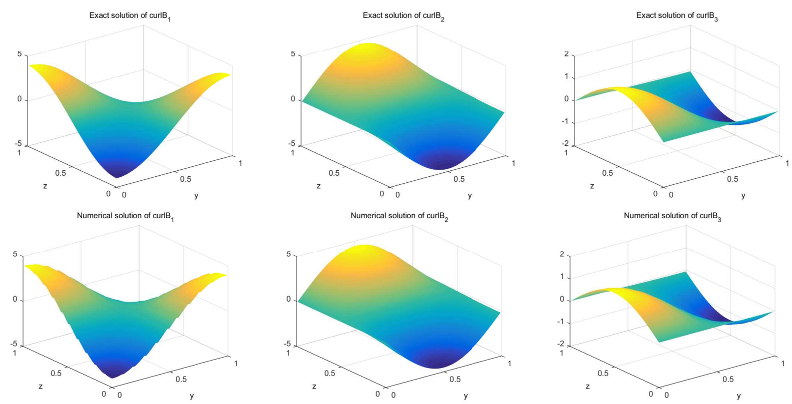

FIGURE 2. The true solution (up) and the numerical solution (down) of three components of $\nabla \times \boldsymbol{B}$ with $\tau = 10^{-5}$ after 100 time steps.

To show our numerical results more intuitively, we list the numerical solutions and the exact solutions of $\boldsymbol{B}$ and curl$\boldsymbol{B}$ by fixing $x = 0.722$ and taking mesh $40 \times 40 \times 40$ in Fig. 1, Fig. 2 . From the Fig. 1 and Fig. 2, we observe that our scheme approximates the exact solutions very well.

6. Conclusions

In this work, we firstly investigate the solvability of the weak formulations of the Magneto-heating problem. This is realized by monotone theory, Arzela-Ascoli theorem and weak convergence analysis under the framework of Rother's method. Furthermore, we also explore the framework of error estimate for the Magneto-heating coupling nonlinear system, which is approached by the boundedness of the α-quench, the L^∞-norm estimate of ξ_h^k, and the Aubin-Nitsche technique. We point that the method can be extended to the higher order time discrete schemes directly to reduce the time step restriction.

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