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A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD

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Abstract. A new finite element method with discontinuous approximation is introduced for solving second order elliptic problem. Since this method combines the features of both conforming finite element method and discontinuous Galerkin (DG) method, we call it conforming DG method. While using DG finite element space, this conforming DG method maintains the features of the conforming finite element method such as simple formulation and strong enforcement of boundary condition. Therefore, this finite element method has the flexibility of using discontinuous approximation and simplicity in formulation of the conforming finite element method. Error estimates of optimal order are established for the corresponding discontinuous finite element approximation in both a discrete H^1 norm and the L^2 norm. Numerical results are presented to confirm the theory.

Key words. Weak Galerkin, discontinuous Galerkin, finite element methods, second order elliptic problem.

1. Introduction

For the sake of clear presentation, we consider Poisson equation with Dirichlet boundary condition in two dimension as our model problem. This conforming DG method can be extended to solve other elliptic problems. The Poisson problem seeks an unknown function u satisfying

$$-\Delta u = f, \quad \text{in } \Omega, \tag{1}$$

$$u = g, \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a polytopal domain in $\mathbb{R}^2$.

Researchers started to use discontinuous approximation in finite element procedure in the early 1970s [2, 7, 12, 17]. Local discontinuous Galerkin methods were introduced in [6]. Then a paper [1] in 2002 provides a unified analysis of discontinuous Galerkin (DG) finite element methods for Poisson equation. Since then, many new finite element methods with discontinuous approximations have been developed such as hybridizable discontinuous Galerkin (HDG) method [5], mimetic finite differences method [10], hybrid high-order (HHO) method [11], virtual element (VE) method [13], weak Galerkin (WG) method [14] and references therein.

The weak form of the problem (1)-(2) is given as follows: find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

The conforming finite element method for the problem (1)-(2) keeps the same simple form as in (3). However, when discontinuous approximation is used, finite element formulations tend to be more complex than (3) to ensure connection of discontinuous function across element boundary. For example, the following is the formulation for the symmetric interior penalty discontinuous Galerkin (IPDG)

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method for the Poisson equation (1) with homogeneous boundary condition: find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$\sum_{T\in\mathcal{T}_h}(\nabla u_h,\nabla v_h)_T-\sum_{e\in\mathcal{E}_h}\int_e\Big(\{\nabla u_h\}[v_h]+\{\nabla v_h\}[u_h]-\alpha h_e^{-1}[u_h][v_h]\Big)=(f,v_h),$$

where α is called a penalty parameter that needs to be tuned.

A first order weakly over-penalized symmetric interior penalty method is proposed in [3] aiming for simplifying the above IPDG formulation by eliminating the two nonsymmetric middle terms: find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$\sum_{T\in\mathcal{T}_h}(\nabla u_h,\nabla v_h)_T+\alpha\sum_{e\in\mathcal{E}_h}h_e^{-3}(\Pi_0[u_h],\ \Pi_0[v_h])_e=(f,v_h),$$

where Π_0 is the L^2 projection to the constant space and α is a positive number. The price paid for a simpler formulation is a worse condition number for the resulting system of linear equations.

In this paper, we propose a new conforming DG method using the same finite element space used in the IPDG method for any polynomial degree $k \geq 1$ but having a simple symmetric and positive definite system: find $u_h \in V_h$ satisfying $u_h = I_h g$ on $\partial\Omega$ and

$$(\nabla_w u_h,\nabla_w v_h)=(f,v_h)\quad\forall v_h\in V_h^0, \tag{4}$$

where ∇_w is called weak gradient introduced in the weak Galerkin finite element method [14, 15]. It follows from (4) that the conforming DG method can be obtained from the conforming formulation simply by replacing ∇ by ∇_w and enforcing the boundary condition strongly. The simplicity of the conforming DG formulation will ease the complexity for implementation of DG methods. The computation of weak gradient $\nabla_w v$ is totally local. Optimal convergence rates for the conforming DG approximation are obtained in a discrete H^1 norm and in the L^2 norm. This new conforming DG method is tested numerically for $k = 1, 2, 3, 4$ and 5, and the results confirm the theory.

2. Finite Element Method

In this section, we will introduce the conforming DG method. For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$.

Let $\mathcal{T}_h$ be a triangulation of the domain Ω with mesh size h that consists of triangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges.

We define the average and the jump on edges for a scalar-valued function v. For an interior edge $e \in \mathcal{E}_h^0$, let T_1 and T_2 be two triangles sharing e. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the two unit outward normal vectors on e, associated with T_1 and T_2, respectively. Define the average $\{\cdot\}$ and the jump $[\cdot]$ on e by

$$\{v\}=\frac{1}{2}(v|_{T_1}+v|_{T_2})\quad\text{and}\quad[v]=v|_{T_1}\mathbf{n}_1+v|_{T_2}\mathbf{n}_2, \tag{5}$$

respectively. If e is a boundary edge, then

$$\{v\}=v,\quad[v]=v\mathbf{n}. \tag{6}$$

For simplicity, we adopt the following notations,

$$\begin{aligned}(v, w) &= (v, w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}(v, w)_T = \sum_{T\in\mathcal{T}_h}\int_T vw d\mathbf{x},\\ \langle v, w\rangle_{\partial\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h}\langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\int_{\partial T} vw ds.\end{aligned}$$

First we define two discontinuous finite element spaces for $k \geq 1$,

$$V_h = \left\{v \in L^2(\Omega) : \ v|_T \in P_k(T), \ \ T \in \mathcal{T}_h\right\}, \tag{7}$$

and

$$V_h^0 = \{v \in V_h : \ v = 0 \text{ on } \partial\Omega\}. \tag{8}$$

Algorithm 2.1. *A conforming DG finite element method for the problem (1)-(2) seeks $u_h \in V_h$ satisfying $u_h = I_h g$ on $\partial\Omega$ and*

$$(\nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} \quad = \quad (f, \ v) \quad \forall v \in V_h^0, \tag{9}$$

where I_h is the kth order Lagrange interpolation.

Next we will discuss how to compute weak gradient $\nabla_w u_h$ and $\nabla_w v$ in (9). The concept of weak gradient ∇_w was first introduced in [14, 15] for weak functions in WG methods and was modified in [16, 8] for the functions in V_h in (7) as follows. For a given $T \in \mathcal{T}_h$ and a function $v \in V_h$, the weak gradient $\nabla_w v \in RT_k(T)$ on T is the unique solution of the following equation,

$$(\nabla_w v, \tau)_T = -(v, \nabla\cdot\tau)_T + \langle\{v\}, \tau\cdot\mathbf{n}\rangle_{\partial T}, \qquad \forall\tau \in RT_k(T), \tag{10}$$

where $RT_k(T) = [P_k(T)]^2 + \mathbf{x}P_k(T)$ and $\{v\}$ is defined in (5) and (6). The weak gradient ∇_w is a local operator computed at each element.

3. Well Posedness

We start this section by introducing two semi-norms $|\!|\!|v|\!|\!|$ and $\|v\|_{1,h}$ for any $v \in V_h$ as follows:

$$|\!|\!|v|\!|\!|^2 \quad = \quad \sum_{T\in\mathcal{T}_h}(\nabla_w v, \nabla_w v)_T, \tag{11}$$

$$\|v\|_{1,h}^2 \quad = \quad \sum_{T\in\mathcal{T}_h}\|\nabla v\|_T^2 + \sum_{e\in\mathcal{E}_h^0} h_e^{-1}\|[v]\|_e^2. \tag{12}$$

The following norm equivalence is proved in Lemma 3.2 [9] with $v_0 = v$ and $v_b = \{v\}$ that there exist two constants C_1 and C_2 independent of h such that

$$C_1\|v\|_{1,h} \leq |\!|\!|v|\!|\!| \leq C_2\|v\|_{1,h}, \quad \forall v \in V_h^0. \tag{13}$$

Lemma 3.1. *The semi-norm $|\!|\!|\cdot|\!|\!|$ defined in (11) is a norm in V_h^0.*

Proof. We only need to prove $v = 0$ if $|\!|\!|v|\!|\!| = 0$ for all $v \in V_h^0$. Let $v \in V_h^0$ and $|\!|\!|v|\!|\!| = 0$. By (13), we have $\|v\|_{1,h} = 0$ which implies that $\nabla v = 0$ in each $T \in \mathcal{T}_h$ and $[v] = 0$ on $e \in \mathcal{E}_h^0$. $\nabla v = 0$ on T implies that v is a constant on each T. $[v] = 0$ on e means that v is continuous. Thus v is a global constant on the whole domain. With $v = 0$ on $\partial\Omega$, we conclude $v = 0$. This completes the proof of the lemma. □

The well posedness of the conforming DG method (9) follows immediately from the above lemma.

4. Error Equation

In this section, we will derive an error equation which will be used in the convergence analysis. First we define $H(\mathrm{div};\Omega)$ space as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e.,

$$H(\mathrm{div};\Omega) = \left\{ \mathbf{v} : \ \mathbf{v} \in [L^2(\Omega)]^2, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}.$$

Define an interpolation operator $\mathbb{Q}_h$ for $\tau \in H(\mathrm{div},\Omega)$ (see [4]) such that $\mathbb{Q}_h\tau \in H(\mathrm{div},\Omega)$, $\mathbb{Q}_h\tau \in RT_k(T)$ on each $T \in \mathcal{T}_h$, and satisfies:

$$(\nabla \cdot \tau, \ v)_T = (\nabla \cdot \mathbb{Q}_h\tau, \ v)_T \qquad \forall v \in P_k(T). \tag{14}$$

Lemma 4.1. *For any $\tau \in H(\mathrm{div},\Omega)$,*

$$-(\nabla \cdot \tau, \ v)_{\mathcal{T}_h} = (\mathbb{Q}_h\tau, \ \nabla_w v)_{\mathcal{T}_h} \quad \forall v \in V_h^0. \tag{15}$$

Proof. Since $\{v\} = v = 0$ on $\partial\Omega$ and $\mathbb{Q}_h\tau \in H(\mathrm{div},\Omega)$, then

$$\langle \mathbb{Q}_h\tau \cdot \mathbf{n}, \{v\}\rangle_{\partial\mathcal{T}_h} = 0. \tag{16}$$

It follows from (14), (10) and (16) that

$$\begin{aligned} -(\nabla \cdot \tau, \ v)_{\mathcal{T}_h} &= -(\nabla \cdot \mathbb{Q}_h\tau, \ v)_{\mathcal{T}_h} \\ &= -(\nabla \cdot \mathbb{Q}_h\tau, \ v)_{\mathcal{T}_h} + \langle \{v\}, \mathbb{Q}_h\tau \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h} \\ &= (\mathbb{Q}_h\tau, \ \nabla_w v)_{\mathcal{T}_h}, \end{aligned}$$

which proves the lemma. □

Define a continuous finite element space $\tilde{V}_h$, a subspace of V_h, by

$$\tilde{V}_h = \{v \in H^1(\Omega) : \ v|_T \in P_k(T), \ \forall T \in \mathcal{T}_h\}. \tag{17}$$

Lemma 4.2. *For any $v \in \tilde{V}_h$,*

$$\nabla_w v = \nabla v.$$

Proof. By the definition of the weak gradient (10) and integration by parts, we have for any $\tau \in RT_k(T)$,

$$\begin{aligned} (\nabla_w v, \tau)_T &= -(v, \nabla \cdot \tau)_T + \langle \{v\}, \tau \cdot \mathbf{n}\rangle_{\partial T} \\ &= -(v, \nabla \cdot \tau)_T + \langle v, \tau \cdot \mathbf{n}\rangle_{\partial T} \\ &= (\nabla v, \tau)_T, \end{aligned}$$

which implies

$$(\nabla_w v - \nabla v, \tau)_T = 0, \quad \forall \tau \in RT_k(T).$$

Since $\nabla_w v - \nabla v \in RT_k(T)$, letting $\tau = \nabla_w v - \nabla v$ in the above equation gives

$$\|\nabla_w v - \nabla v\|_T^2 = 0,$$

which proves the lemma. □

Let $e_h = I_h u - u_h$. Obviously, $e_h \in V_h^0$. Recall that $I_h u$ is the kth order Lagrange interpolation of u and then $I_h u \in \tilde{V}_h$. By Lemma 4.2, we have

$$\nabla_w I_h u \ = \ \nabla I_h u. \tag{18}$$

Lemma 4.3. *Let $e_h = I_h u - u_h$ be the error of the finite element solution arising from (9). Then we have*

$$(\nabla_w e_h,\ \nabla_w v)_{\mathcal{T}_h} \quad = \quad l(u,v), \quad \forall v \in V_h^0, \tag{19}$$

where

$$l(u,v) \quad = \quad (\nabla I_h u - \mathbb{Q}_h \nabla u,\ \nabla_w v)_{\mathcal{T}_h}. \tag{20}$$

Proof. Testing the equation (1) by $v \in V_h^0$ gives

$$-(\nabla \cdot \nabla u, v) = (f, v). \tag{21}$$

It follows from (15) that

$$(\mathbb{Q}_h \nabla u, \nabla_w v)_{\mathcal{T}_h} \quad = \quad (f, v). \tag{22}$$

Adding $(\nabla_w I_h u, \nabla_w v)_{\mathcal{T}_h}$ to the both sides of the equation (22) and using (18) yield

$$(\nabla_w I_h u, \nabla_w v)_{\mathcal{T}_h} \quad = \quad (f, v) + (\nabla I_h u - \mathbb{Q}_h \nabla u, \nabla_w v)_{\mathcal{T}_h}. \tag{23}$$

The difference of (23) and (9) gives (19). We have proved the lemma. □

5. Error Estimates

In this section, we shall establish optimal order error estimates for u_h in a discrete H^1 norm and the L^2 norm.

5.1. An Estimate in a Discrete H^1 Norm.

We start this subsection by bounding the term $l(u,v)$ defined in (20).

Lemma 5.1. *Let $u \in H^{k+1}(\Omega)$ and $v \in V_h^0$. Then, the following estimate holds,*

$$|l(u,v)| \quad \le \quad Ch^k |u|_{k+1} |||v|||. \tag{24}$$

Proof. Using the Cauchy-Schwarz inequality and the definition of I_h and $\mathbb{Q}_h$, we have

$$\begin{aligned}
l(u,v) &= (\nabla I_h u - \mathbb{Q}_h(\nabla u), \nabla_w v)_{\mathcal{T}_h} \\
&\le \sum_{T\in\mathcal{T}_h} \|\nabla I_h u - \mathbb{Q}_h(\nabla u)\|_T \|\nabla_w v\|_T \\
&\le \Big(\sum_{T\in\mathcal{T}_h} \|\nabla I_h u - \mathbb{Q}_h(\nabla u)\|_T^2\Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\nabla_w v\|_T^2\Big)^{1/2} \\
&\le \Big(\sum_{T\in\mathcal{T}_h} \|\nabla I_h u - \nabla u\|_T^2 + \|\nabla u - \mathbb{Q}_h(\nabla u)\|_T^2\Big)^{1/2} |||v||| \\
&\le Ch^k |u|_{k+1} |||v|||,
\end{aligned}$$

which proves the lemma. □

Theorem 5.1. *Let $u_h \in V_h$ be the finite element solution of (9). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$|||u_h - I_h u||| \le Ch^k |u|_{k+1}. \tag{25}$$

Proof. Letting $v = e_h$ in (19) gives

$$\|\!|e_h|\!\|^2 = l(u, e_h). \tag{26}$$

Using (24), we arrive

$$\|\!|e_h|\!\|^2 \le Ch^k |u|_{k+1} \|\!|e_h|\!\|,$$

which completes the proof. □

5.2. An Estimate in the L^2 Norm. In this subsection, we will derive the error estimate for u_h in the L^2 norm. First we define $\tilde{V}_h^0$ a subspace of $\tilde{V}_h$ in (17) as

$$\tilde{V}_h^0 = \{v \in \tilde{V}_h : v|_{\partial\Omega} = 0\}. \tag{27}$$

Let $\tilde{u}_h \in \tilde{V}_h$ be the conforming finite element solution such that $\tilde{u}_h = I_h g$ on $\partial\Omega$ and satisfies

$$(\nabla \tilde{u}_h, \nabla v) = (f, v) \quad \forall v \in \tilde{V}_h^0. \tag{28}$$

Since $\tilde{V}_h^0 \subset V_h^0$, by Lemma 4.2, (9) and (28), we have

$$(\nabla_w u_h - \nabla \tilde{u}_h, \nabla v) = 0, \quad \forall v \in \tilde{V}_h^0. \tag{29}$$

Consider the dual problem: seek $\Phi \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot (\nabla \Phi) = u_h - \tilde{u}_h \quad \text{in } \Omega. \tag{30}$$

Assume that the following H^2-regularity holds

$$\|\Phi\|_2 \le C\|u_h - \tilde{u}_h\|. \tag{31}$$

Now we are ready to derive the L^2 error estimate.

Theorem 5.2. *Let $u_h \in V_h$ be the finite element solution of (9). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and that (31) holds true. Then, there exists a constant C such that*

$$\|u - u_h\| \le Ch^{k+1}|u|_{k+1}. \tag{32}$$

Proof. By the triangle inequality, we have

$$\|u - u_h\| \le \|u - \tilde{u}_h\| + \|u_h - \tilde{u}_h\|. \tag{33}$$

The definition of $\tilde{u}_h$ implies

$$\|u - \tilde{u}_h\| \le Ch^{k+1}|u|_{k+1}. \tag{34}$$

Next we will estimate $\|u_h - \tilde{u}_h\|$. Let $\Phi_h \in V_h^0$ be the conforming DG approximation to the problem (30) satisfying

$$(\nabla_w \Phi_h, \nabla_w v) = (u_h - \tilde{u}_h, v), \quad \forall v \in V_h^0. \tag{35}$$

Letting $v = u_h - \tilde{u}_h \in V_h^0$ in (35) and using Lemma 4.2 and (29), we have,

$$\begin{aligned}\|u_h - \tilde{u}_h\|^2 &= (\nabla_w \Phi_h, \nabla_w (u_h - \tilde{u}_h))_{\mathcal{T}_h} = (\nabla_w \Phi_h, \nabla_w u_h - \nabla \tilde{u}_h)_{\mathcal{T}_h} \\ &= (\nabla_w (\Phi_h - I_h \Phi), \nabla_w u_h - \nabla \tilde{u}_h)_{\mathcal{T}_h}.\end{aligned}$$

By the Cauchy-Schwartz inequality, (25) and (31), then

$$\begin{aligned}\|u_h - \tilde{u}_h\|^2 &\le \|\!|\Phi_h - I_h \Phi|\!\| \left(\|\!|u_h - I_h u|\!\| + \|\nabla (I_h u - \tilde{u}_h)\|\right) \\ &\le Ch|\Phi|_2 h^k |u|_{k+1} \\ &\le Ch^{k+1}|u|_{k+1}\|u_h - \tilde{u}_h\|,\end{aligned}$$

which implies

$$\|u_h - \tilde{u}_h\| \le Ch^{k+1}|u|_{k+1}. \tag{36}$$

Combining (34) and (36) with (33), we have proved the theorem. □

6. Numerical Example

We solve the following Poisson equation on the unit square:

$$-\Delta u = 2\pi^2 \sin \pi x \sin \pi y, \quad (x, y) \in \Omega = (0, 1)^2, \tag{37}$$

with the boundary condition $u = 0$ on $\partial\Omega$.

In computation, the first grid consists of two unit right triangles cutting from the unit square by a forward slash. The high level grids are the half-size refinement of the previous grid. We apply P_k finite element methods V_h and list the error and the order of convergence in the following table. The numerical results confirm the convergence theory.

TABLE 1. Error profiles and convergence rates for (37).

level	$\|u_h - u\|$	rate	$\|\|\|u_h - I_h u\|\|\|$	rate
	by P_1 elements			
6	0.7280E-03	2.09	0.7199E-01	0.91
7	0.1751E-03	2.06	0.3718E-01	0.95
8	0.4287E-04	2.03	0.1890E-01	0.98
	by P_2 elements			
6	0.6446E-05	2.94	0.1744E-02	1.95
7	0.8197E-06	2.98	0.4424E-03	1.98
8	0.1033E-06	2.99	0.1113E-03	1.99
	by P_3 elements			
6	0.4457E-07	4.02	0.2293E-04	2.97
7	0.2772E-08	4.01	0.2902E-05	2.98
8	0.1730E-09	4.00	0.3650E-06	2.99
	by P_4 elements			
5	0.2057E-07	5.03	0.4748E-05	3.95
6	0.6344E-09	5.02	0.3009E-06	3.98
7	0.1984E-10	5.00	0.1893E-07	3.99
	by P_5 elements			
4	0.2481E-07	6.04	0.3223E-05	4.94
5	0.3811E-09	6.02	0.1024E-06	4.98
6	0.5938E-11	6.00	0.3225E-08	4.99

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