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ANALYSIS OF A SECOND-ORDER DECOUPLED TIME-STEPPING SCHEME FOR TRANSIENT VISCOELASTIC FLOW

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Abstract. In this paper, we propose and analyze a decoupled second order backward difference formula (BDF2) time-stepping algorithm for solving transient viscoelastic fluid flow. The spatial discretization is based on continuous Galerkin finite element approximation for the velocity and pressure, and discontinuous Galerkin finite element approximation for the viscoelastic stress tensor. To obtain a non-iterative decoupled algorithm from the fully discrete nonlinear system, we employ a second order extrapolation in time to the nonlinear terms. The algorithm requires the solution of one Navier-Stokes problem and one constitutive equation per time step. For mesh size h and temporal step size Δt sufficiently small satisfying $\Delta t \leq Ch^{d/4}$, a priori error estimates in terms of Δt and h are derived. Numerical tests are presented that illustrates the accuracy and stability of the algorithm.

Key words. Viscoelasticity, finite element method, discontinuous Galerkin method, decoupled scheme, error estimates, BDF2.

1. Introduction

Time accurate computation of viscoelastic flows are important in many engineering applications involving non-Newtonian fluid mechanics, see [13, 17, 21]. The Oldroyd-B model is one of the simplest constitutive models capable of describing the viscoelastic behavior of flows in which the extra stress tensor is defined by a hyperbolic partial differential equation. The challenges posed by the hyperbolic character of the equation for the extra stress tensor such as spurious oscillations warrants care in discretizing this equation. For the steady state problem, a discontinuous Galerkin (DG) finite element approximation of the constitutive equation was proposed and analyzed in [2]. In [16], a decoupled algorithm was analyzed for efficient implementation of the scheme discussed in [2]. In [20], a Streamline Upwind Petrov Galerkin (SUPG) approximation was employed to discretize the constitutive equation and an error analysis was presented. For the unsteady problem, a DG discretization based approximation for the constitutive equation in inertialess flow was studied in [3]. In [5], a fractional step θ method for time integration, combined with Taylor-Hood finite element and the SUPG spatial discretization is presented. An implicit backward Euler time discretization and continuous piecewise linear finite element in space for three field S-tokes problem is discussed in [1]. In [22], unconditional error estimates of finite element approximation to the viscoelastic flows, with DG discretization for the constitutive equation is discussed. With first order implicit Euler

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temporal discretization and Taylor-Hood finite element approximation for the velocity and pressure, they derived error estimates under the assumption $\Delta t \leq Ch^{3/2}$. In [9], a first order implicit Euler time discretization and SUPG discretization for the constitutive equation was discussed and error estimates were derived under the assumption that $\Delta t, \nu < C\, h^{d/2}$, where ν is the stabilization parameter of SUPG method. In [8], a Crank-Nicolson time discretization scheme with a DG approximation for the constitutive equation presented and error estimates were derived under the assumption that $\Delta t \leq Ch^{d/4}$.

In this paper, we propose and analyze a partitioned time stepping scheme for the viscoelastic flow model based on second order backward Euler time discretization. A second order in time extrapolation is used to effect a decoupling of the subphysics problems and to have the approximation determined at each time level by the solution of a single linear system. With finite element approximation of the momentum equation and DG method for the constitutive equation, we derive error estimates under the assumption $\Delta t \leq Ch^{d/4}$.

The rest of the paper is organized as follows: In Section 2, we introduce the decoupled second-order backward difference time stepping scheme assuming mixed finite element spatial discretizations for the time dependent viscoelastic flow with constitutive equation stabilized by discontinuous Galerkin (DG) approximation. In §3, we present the error estimates for the fully discrete approximations. In §4, we present numerical results that illustrate the accuracy and efficiency of our algorithm. We close by providing some remarks in §5.

2. The Oldroyd B model and decoupled time-stepping scheme

2.1. The Oldroyd B model. We consider a fluid flow in a bounded domain Ω in $\mathbb{R}^d$, $(d = 2, 3)$ with Lipschitzian boundary Γ. Let p denotes the pressure, $\mathbf{u}$ the velocity, $\mathbb{D}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$ the rate of strain tensor and σ_{tot} the total stress tensor. An Oldroyd's model of differential type with a single relaxation time is obtained by setting $\sigma_{\text{tot}} = -pI + \sigma + \sigma_N$ where σ is the viscoelastic part of the extra stress tensor and $\sigma_N = 2(1-\alpha)\mathbb{D}(\mathbf{u})$ is the Newtonian part, $1 < \alpha \leq 1$. The Oldroyd-B model of viscoelastic flow then is the following

(1)
$$\begin{cases} \partial_t \mathbf{u} - \dfrac{2(1-\alpha)}{Re} \nabla \cdot \mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{Re}\nabla p \; - \; \nabla\cdot\sigma = \mathbf{f} \quad \text{in } \Omega \times (0,T] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \nabla\cdot\mathbf{u} \; = \; 0 \quad \text{in } \Omega\times(0,T] \\ \partial_t\sigma + (\mathbf{u}\cdot\nabla)\sigma - \frac{2\alpha}{\lambda}\mathbb{D}(\mathbf{u}) + g_a(\sigma,\nabla\mathbf{u}) + \frac{\sigma}{\lambda} \; = \; 0 \quad \text{in } \Omega\times(0,T] \end{cases}$$

where the function $\mathbf{f}$ is the external force and the function g_a is defined by

$$g_a(\sigma, \nabla\mathbf{u}) := \frac{1-a}{2}(\sigma\nabla\mathbf{u} + (\nabla\mathbf{u})^t\sigma) - \frac{1+a}{2}((\nabla\mathbf{u})\sigma + \sigma(\nabla\mathbf{u})^t)\,,$$

where $a \in [-1,1]$. Moreover T (>0) denotes time, Re the Reynolds number and λ the Weissenberg number.

The solution of (1) is required to satisfy boundary conditions. For the velocity, we set $\mathbf{u} = \mathbf{g}$ on Γ. Due to the hyperbolic character of constitutive equation for the stress σ for fixed $\mathbf{u}$, we need to apply $\sigma = \widehat{\sigma}$ on the inflow boundary $\Gamma^- = \{\mathbf{x} \in \Gamma \ : \ \mathbf{u}\cdot\mathbf{n} < 0\}$. In order to simplify the analysis, we assume that $\mathbf{g} = \mathbf{0}$ which implies that there is no boundary condition necessary for the stress σ. The initial conditions are prescribed as

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{and} \quad \sigma(\mathbf{x},0) = \sigma_0(\mathbf{x}) \quad \text{in } \Omega \,.$$

2.2. The weak form. In this section, we define the weak form of (1). The following notation will be employed. For integer $k \ge 0$, $C^k(\Omega)$ denotes the space of functions k times continuously differentiable in Ω and the space $C^k(\overline{\Omega})$ denotes the functions in $C^k(\Omega)$ bounded and uniformly continuous in Ω with derivatives up to the k^{th}-order, and the space $C^{k,1}(\overline{\Omega})$ consists of functions in $C^k(\overline{\Omega})$ that are Lipschitz-continuous in $\overline{\Omega}$ with derivatives up the k^{th}-order. For a Banach space X, we denote by $L^p(0,T;X)$ the time-space function space endowed with the norm $\|w\|_{L^p(0,T;X)} := \left(\int_0^T \|w\|_X^p \, dt\right)^{1/p}$ if $1 \le p < \infty$ and $\operatorname*{ess\,sup}_{t\in[0,T]} \|w\|_X$ if $p = \infty$. We will often use the abbreviated notation $L^p(X) := L^p(0,T;X)$ for convenience. The symbol $C([0,T];X)$ denotes the set of continuous functions $u : [0,T] \to X$ endowed with the norm $\|u\|_{C(0,T;X)} := \max_{0\le t\le T} \|u(t)\|_X$. For any integer $k \ge 1$, let $W^{k,m}(\Omega)$ be the Sobolev space of functions in $L^p(\Omega)$ with derivatives up-to the k^{th}-order endowed with the norm $\|\phi\|_{k,m} := \left[\sum_{|\alpha|\le k} \int_\Omega |\partial_x^\alpha \phi(\mathbf{x})|^m dx\right]^{\frac{1}{m}}$ where $\partial_x^\alpha \phi(\mathbf{x}) := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}}\phi(\mathbf{x})$, $\alpha := (\alpha_1,\cdots,\alpha_d)$, $\alpha_i \ge 0$, $|\alpha| := \sum_{i=1}^d \alpha_i$. We denote by $H^k(\Omega)$ the space $W^{k,2}(\Omega)$, when $m = 2$, and drop the subscripts $p(=2)$ in referring to the norm in $H^k(\Omega)$. Moreover, we will use the following simplified norm notations:

$$\|u\| := \|u\|_{L^2(\Omega)} \quad \text{and} \quad \|u\|_\infty := \|u\|_{L^\infty(\Omega)} \,.$$

We introduce the time discrete space $l^p(Z)$ associated with $L^p(0,T;Z)$; $l^p(Z)$ is the space of Z-valued sequences $w := \{w_n; n = 1,\ldots,N\}$ with norm $\|\cdot\|_{l^p(Z)}$ defined by

$$\|w\|_{l^p(Z)} := \begin{cases} (\Delta t \displaystyle\sum_{n=1}^N \|w_n\|_Z^p)^{1/p} & \text{if } 1 \le p < \infty \\ \displaystyle\max_{1\le n\le N} \|w_n\|_Z & \text{if } p = \infty \,. \end{cases}$$

We will also use the following spaces

$$\mathbf{H}_0^1(\Omega) := \{\mathbf{u} \in \mathbf{H}^1(\Omega) \ : \ \mathbf{u}|_\Gamma = \mathbf{0}\ \},$$

$$S := \{\tau = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji}\,,\ \tau_{ij} \in L^2(\Omega)\,,\ 1 \le i,j \le d\,\} \cap \{\tau = (\tau_{ij}) \,|\, \mathbf{u}\cdot\nabla\tau \in L^2(\Omega)\,,\ \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)\,\},$$

$$L_0^2(\Omega) := \{p \in L^2(\Omega) \ : \ \int_\Omega p\, d\Omega = 0\ \}$$

and

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \ : \quad b(\mathbf{v}, q) = 0\,, \quad \forall q \in L_0^2(\Omega)\}\,.$$

For later purposes, we recall Korn's inequality (see [14])

$$(\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{v})) \ge \lambda_k \|\mathbf{v}\|_1^2 \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega)\,,$$

the Poincaré inequality

$$\|\mathbf{v}\|^2 \le \lambda_p \|\nabla\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,,$$

the Gagliardo-Nirenberg interpolation inequality [7]

$$\|\mathbf{u}\|_{L^q(\Omega)} \le C \|\nabla\mathbf{u}\|_{L^p(\Omega)}^{\lambda} \|\mathbf{u}\|_{L^r(\Omega)}^{1-\lambda}$$

for $0 \le \lambda \le 1$ and $\frac{1}{q} = \lambda(\frac{1}{p} - \frac{1}{d}) + (1-\lambda)\frac{1}{r}$ and the Agmon's inequality

$$\|\mathbf{u}\|_\infty \le C \|\mathbf{u}\|_1^{\frac{1}{2}} \|\mathbf{u}\|_2^{\frac{1}{2}} \quad \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)\,.$$

We define the following bilinear and trilinear forms given by

$$a(\mathbf{u}, \mathbf{v}) := \frac{2(1-\alpha)}{Re} \int_\Omega \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u})\, d\Omega\,,$$

$$\begin{aligned} c(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad &:= \quad \frac{1}{2}\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}] d\Omega \\ &= \quad \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}]\, d\Omega \\ &= \quad -\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}]\, d\Omega\,, \end{aligned}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ with $(\mathbf{u}\cdot\mathbf{n})\mathbf{v}\cdot\mathbf{w} = 0$ on Γ, and

$$b(\mathbf{v}, r) := -\int_\Omega r\, \nabla\cdot\mathbf{v}\, d\Omega \qquad \text{for } (\mathbf{v}, r) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)\,.$$

A weak formulation of the problem (1) is derived by multiplying (1) by test functions and integrating by parts.

Definition 2.1 *For a given* $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ *a triple* $(\mathbf{u}, \sigma, p) \in L^2(0,T;\mathbf{H}_0^1(\Omega)) \times L^2(0,T;S) \times L^2(0,T;L_0^2(\Omega))$ *with* $(\partial_t\mathbf{u}, \partial_t\sigma) \in L^1(0,T;\mathbf{H}^{-1}(\Omega)) \times L^1(0,T;S')$ *is said to be a weak solution of (1) if*

(2)

$$\left\{ \begin{array}{rcl} (\partial_t\mathbf{u}, \mathbf{v}) & + & a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \frac{1}{Re}(\sigma, \mathbb{D}(\mathbf{v})) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \\ b(\mathbf{u}, q) & = & 0 \\ (\partial_t\sigma, \tau) & + & \frac{1}{\lambda}(\sigma, \tau) - \frac{2\alpha}{\lambda}(\mathbb{D}(\mathbf{u}), \tau) + (\mathbf{u}\cdot\nabla\sigma, \tau) + (g(\sigma, \nabla\mathbf{u}), \tau) = 0 \end{array} \right.$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $q \in L_0^2(\Omega)$ and $\tau \in S$. For existence and stability results for the solutions of problem (1) and the corresponding creeping flow problem (neglecting the inertial terms) the readers are referred to [11, 18].

2.3. The partitioned time-stepping scheme. In this section, we formulate a partitioned time stepping scheme for viscoelastic flow model and derive error estimates for the fully discrete scheme by assuming finite element spatial discretization. We begin by describing the finite element spatial discretization and summarizing approximation properties used in the subsequent analysis.

We assume the domain Ω is a convex polyhedron, for simplicity, and partition Ω into a mesh $\mathcal{T}_h$ with $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$ so that $diameter(K) \leq h$ and any two closed elements K_1 and $K_2 \in \mathcal{T}_h$ are either disjoint or share exactly one face, side or vertex. Suppose further that $\mathcal{T}_h$ is a shape regular and quasi-uniform triangulation. To discretize the Oldroyd B model in space by the finite element method, we select finite element spaces

$$\text{velocity} \; : \; \mathbf{X}_h \subset \mathbf{H}_0^1(\Omega), \;\; \text{pressure} \; : \; Q_h \subset L_0^2(\Omega), \; \text{stress} \; : \; S_h \subset S \,,$$

where

$$\begin{aligned}
\mathbf{X}_h :=& \{\mathbf{v} \in \mathbf{C}(\overline{\Omega}) \; : \; \mathbf{v}|_K \in \mathcal{P}_k(K)\,, \forall K \in \mathcal{T}_h\}\,,\\
S_h :=& \{\sigma \; : \; \sigma|_K \in \mathcal{P}_q(K)\,, \forall K \in \mathcal{T}_h\}\,,\\
Q_h :=& \{p \in C(\overline{\Omega}) \; : \; p|_K \in \mathcal{P}_{k-1}(K)\,, \forall K \in \mathcal{T}_h\}\,,\\
\mathbf{V}_h :=& \{\mathbf{v}_h \in X_h \; : \; b(\mathbf{v}_h, r_h) = 0 \quad \forall r_h \in Q_h \;\}\,,
\end{aligned}$$

and $\mathcal{P}_k$ is the space of polynomials of degree less or equal to k on $K \in \mathcal{T}_h$, see [6] for details concerning such finite element discretizations. We assume that $(\mathbf{X}_h, Q_h, S_h)$ satisfies the following approximation properties: for $(\mathbf{w}, r, \tau) \in \mathbf{H}^{k+1}(\Omega) \times H^k(\Omega) \times H^{q+1}(\Omega)$, we have that there exists interpolants $(\pi_h\mathbf{w}, \pi_h r) \in \mathbf{X}_h \times Q_h$ and $\pi_h\tau \in S_h$ such that

$$\begin{aligned}
\|\mathbf{w} - \pi_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - \pi_h\mathbf{w})\| &\leq C\, h^{k+1}\|\mathbf{w}\|_{k+1}\,,\\
\|r - \pi_h r\| &\leq C\, h^k\|r\|_k\,,
\end{aligned}$$

and

$$\|\tau - \pi_h\tau\| + h\|\nabla(\tau - \pi_h\tau)\| \leq C\, h^{q+1}\|\tau\|_{q+1}\,.$$

The finite dimensional subspaces are assumed to satisfy the so called inverse inequality [4]: For any integers l and m $(0 \leq l \leq m \leq 1)$ and any real numbers p and q $(1 \leq p \leq q \leq \infty)$ it holds that

$$\|\psi_h\|_{m,q} \leq ch^{l-m+d(1/q-1/p)}\|\psi_h\|_{l,p} \quad \forall\psi_h \in \mathbb{X}_h\,. \tag{3}$$

Moreover, we assume that the fluid velocity and pressure spaces $\mathbf{X}_h$ and Q_h satisfy the following discrete inf-sup condition necessary for stability [10]:

$$\inf_{q_h\in Q_h} \sup_{\mathbf{v}_h\in\mathbf{X}_h} b(\mathbf{v}_h, q_h) \geq \beta > 0\,.$$

We will employ the discontinuous finite element method to discretize the constitutive equation. To this end, following [2], we introduce $\partial\, K^-(\mathbf{u}) := \{\mathbf{x} \in \partial\, K\,, \mathbf{u}\cdot\mathbf{n} < 0\,\}$, where ∂K is the boundary of $K \in \mathcal{T}_h$ and $\mathbf{n}$ is the outward unit normal to ∂K, and $\overset{+}{\tau^{-}}(\mathbf{u})(\mathbf{x}) := \lim_{\epsilon \to 0^{\pm}} \tau(\mathbf{x} + \epsilon \mathbf{u})$. Moreover, we define

$$(\sigma, \tau)_h := \sum_{K \in \mathcal{T}_h} (\sigma, \tau)_K ,$$

$$\langle \overset{+}{\sigma^{-}}, \overset{+}{\tau^{-}} \rangle_{h,u} := \sum_{K \in \mathcal{T}_h} \int_{\partial K^{-}(\mathbf{u})} (\overset{+}{\sigma^{-}}(\mathbf{u}), \overset{+}{\tau^{-}}(\mathbf{u})) |\mathbf{u} \cdot \mathbf{n}| \, ds$$

$$\langle\langle \overset{+}{\sigma^{-}} \rangle\rangle^2_{h,u} := \langle \overset{+}{\sigma^{-}}, \overset{+}{\sigma^{-}} \rangle_{h,u} , \quad \|\tau\|_{0,\Gamma^h} := (\sum_{K \in \mathcal{T}_h} |\tau|^2_{0,\partial K})^{1/2} .$$

We approximate the convection term $(\mathbf{u} \cdot \nabla \sigma, \tau)$ by means of an operator B on $\mathbf{X}_h \times S_h \times S_h$ defined by

$$\begin{aligned} B(\mathbf{u}, \tau, \sigma) \; &:= \; (\mathbf{u} \cdot \nabla \tau, \sigma)_h + \tfrac{1}{2}(\nabla \cdot \mathbf{u}\tau, \sigma) + < \tau^{+} - \tau^{-}, \sigma^{+} >_{h,u} \\ &= \; -(\mathbf{u} \cdot \nabla \sigma, \tau) - \tfrac{1}{2}(\nabla \cdot \mathbf{u}\sigma, \tau) + < \tau^{-}, \sigma^{-} - \sigma^{+} >_{h,u} , \end{aligned}$$

see [15]. The last equality implies that

$$B(\mathbf{u}, \tau, \tau) = \frac{1}{2} \langle\langle \tau^{+} - \tau^{-} \rangle\rangle^2_{h,u} .$$

Further, we divide the time interval $[0, T]$ into N subintervals $[t_n, t_{n+1}]$ $(n = 0, 1, 2, \ldots, N-1)$, satisfying

$$0 < t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T .$$

Let $\Delta t := t_n - t_{n-1}$ be the time step and let $\phi^n(\cdot)$ be a given algorithmic approximation to $\phi(\cdot, t_n)$. Let $\mathcal{D}(\phi^{n+1})$ denote the 2-step backward difference operator

$$\mathcal{D}(\phi^n) := \frac{3\phi^n - 4\phi^{n-1} + \phi^{n-2}}{2\Delta t}$$

and $\mathcal{I}(\phi^{n+1/2})$ denote the extrapolation operator $\mathcal{I}(\phi^n) := 2\phi^{n-1} - \phi^{n-2}$.

Based on the weak form (2), the proposed decoupled time stepping scheme is as follows.

ALGORITHM 2.1. *Given* $(\mathbf{u}_h^i, p_h^i, \sigma_h^i) \in \mathbf{X}_h \times Q_h \times S_h$, $i = 0, 1$, *find* $\{(\mathbf{u}_h^n, p_h^n, \sigma_h^n) \in \mathbf{X}_h \times Q_h \times S_h$ *such that*

$$(4) \quad \begin{cases} (\mathcal{D}\mathbf{u}_h^n, \mathbf{v}_h) & + \quad a(\mathbf{u}_h^n, \mathbf{v}_h) + c(\mathcal{I}(\mathbf{u}_h^n), \mathbf{u}_h^n, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^n) \\ & + \quad \frac{1}{Re}(\mathcal{I}(\sigma_h^n), \mathbb{D}(\mathbf{v}_h)) = (\mathbf{f}^n, \mathbf{v}_h) , \\ b(\mathbf{u}_h^n, r_h) & = \quad 0 , \\ (\mathcal{D}\sigma_h^n, \tau_h) & + \quad \frac{1}{\lambda}(\sigma_h^n, \tau_h) + B(\mathcal{I}(\mathbf{u}_h^n), \sigma_h^n, \tau_h) - \frac{2\alpha}{\lambda}(\mathbb{D}(\mathbf{u}_h^n), \tau_h) \\ & + \quad (g_a(\mathcal{I}(\sigma_h^n), \nabla \mathbf{u}_h^n), \tau_h) = 0 , \end{cases}$$

$\forall (\mathbf{v}_h, \tau_h, r_h) \in \mathbf{X}_h \times S_h \times Q_h$, *for* $n = 2, \ldots, N$. Algorithm 2.1 employs a two-step BDF2 discretization for the time derivative terms. A two-step extrapolation in time is used to uncouple the system into two subproblem solves. We note that the method is decoupled but sequential: $\sigma_h^{n-1} \to \mathbf{u}_h^n \to \sigma_h^n$.

3. Error analysis

In this section, we discuss the accuracy and convergence of the scheme. To this end, we assume that the exact solution satisfies the following.
Assumption A1. The exact solution $(\mathbf{u}, p, \sigma)$ of (1) satisfy

$$\mathbf{u} \in C([0,T]; \mathbf{V} \cap \mathbf{W}^{1,\infty}) \cap H^1(0,T; \mathbf{H}^{k+1}(\Omega)) \cap H^3(0,T; \mathbf{L}^2(\Omega))\,,$$
$$\sigma \in C([0,T]; S \cap W^{1,\infty}) \cap H^1(0,T; H^{q+1}(\Omega)) \cap H^3(0,T; L^2(\Omega))\,,$$
$$p \in C([0,T]; L^2_0(\Omega) \cap H^k(\Omega))\,.$$

We will also use the following induction hypothesis in the sequel:
Assumption A2 (Induction Hypothesis). The approximate solutions $\mathbf{u}_h^{n-1}$ and σ_h^{n-1} satisfy

$$\|\mathbf{u}_h^{n-1}\|_\infty\,,\ \|\sigma_h^{n-1}\|_\infty \le K\,.$$

We define the Stokes projection as follows. Given $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$, we define the Stokes projection $(\underline{\mathbf{u}}_h, \underline{p}_h) \in \mathbf{X}_{h,g_h} \times Q_h$ as the solution of the problem

$$\begin{aligned} a(\mathbf{u} - \underline{\mathbf{u}}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - \underline{p}_h) &= 0 \quad \forall \mathbf{v}_h \in \mathbf{X}_h\,, \\ b(\mathbf{u} - \underline{\mathbf{u}}_h, r_h) &= 0 \quad \forall r_h \in Q_h\,. \end{aligned} \tag{5}$$

Using the H^2- regularity property of the Stokes operator in smooth domains and a duality argument, we can show the following approximation property holds:

$$\|\mathbf{u} - \underline{\mathbf{u}}_h\|_1 + \|p - \underline{p}_h\| \le ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)\,. \tag{6}$$

Recall Gagliardo-Nirenberg's interpolation inequality yields

$$\|\phi\|_{0,\infty} + \|\phi\|_{1,3} \le C\|\phi\|_1^{\frac{1}{2}}\|\phi\|_2^{\frac{1}{2}}\,.$$

This together with H^2-regularity of the Stokes operator in regular domains yields

$$\|\underline{\mathbf{u}}_h\|_\infty + \|\underline{\mathbf{u}}_h\|_{1,3} \le c(\|\mathbf{u}\|_2 + \|p\|_1)\,. \tag{7}$$

We also need to estimate the following two quantities in error analysis: First by Taylor expansion with integral remainder and by Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\|\partial_t\phi(t_n) - \mathcal{D}(\phi(t_n))\| \\ \le\ &\Big\|\frac{1}{2\Delta t}\int_{t_{n-2}}^{t_n}\left\{2(t-t_{n-1})_+^2 - \frac{1}{2}(t-t_{n-2})^2\right\}\partial_t^3\phi\,dt\Big\| \\ \le\ &C(\Delta t)^{3/2}\|\partial_t^3\phi(t)\|_{L^2(t_{n-2},t_n;L^2(\Omega))}\,, \end{aligned} \tag{8}$$

where $(t-t_{n-1})_+ = \max\left((t-t_{n-1}), 0\right)$. Similarly, for the extrapolation operator $\mathcal{I}(\phi^n)$, we can show

$$\|\mathcal{I}(\phi^n) - \phi^n\|_{H^k} \le C(\Delta t)^{3/2}\|\partial_t^2\phi(t)\|_{L^2(t_{n-2},t_n;H^k)}\,. \tag{9}$$

We also cite a discrete Grönwall lemma which is useful in our analysis in the sequel.

Lemma 3.1 (Discrete Grönwall lemma [12]**)** Let d, Δt, $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$, and $\{d_n\}_{n\geq 0}$ be nonnegative numbers such that

$$a_n + \Delta t \sum_{i=0}^{n} b_i \leq \Delta t \sum_{i=0}^{n} d_i a_i + \Delta t \sum_{i=0}^{n} c_i + d,$$

for $n \geq 0$. Suppose that $\Delta t d_i < 1$ for all i. Then

$$a_n + \Delta t \sum_{i=0}^{n} b_i \leq \exp\left(\Delta t \sum_{i=0}^{n} \frac{d_i}{1 - \Delta t d_i} \right) \left(\Delta t \sum_{i=0}^{n} c_i + d \right),$$

for all $n \geq 0$. A proof of this result can be found, for e.g, in [12].

Under the preparation above, we can obtain the following error estimate for velocity and stress tensor.

Theorem 3.2 Suppose that the assumption (A1) and (A2) hold with a positive number h_0 and a positive integer k and the initial conditions $(\mathbf{u}_h^i, \sigma_h^i)\,, i = 0, 1$ satisfy

$$\sum_{i=0}^{1} \|\mathbf{u}_h^i - \mathbf{u}^i\|^2 + \|\sigma_h^i - \sigma^i\|^2 + \Delta t \|\mathbb{D}(\mathbf{u}_h^i - \mathbf{u}^i)\|^2 \leq c(h^{2k} + h^{2q})\,.$$

In addition, assume that $\Delta t \leq c\, h^{d/2}$. Then, for $l = 2, \ldots, N$ we have the following error estimates

$$\|\mathbf{u}^l - \mathbf{u}_h^l\|^2 + \Delta t \sum_{n=0}^{l} \|\mathbb{D}(\mathbf{u}^l - \mathbf{u}_h^l)\|^2 \leq C(\Delta t^4 + h^{2k} + h^{2q})$$

and

$$\|\sigma^l - \sigma_h^l\|^2 + \Delta t \sum_{n=0}^{l} \|\sigma^l - \sigma_h^l\|^2 \leq C(\Delta t^4 + h^{2k} + h^{2q})$$

for some constant C independent of the mesh size h and time step Δt.

Proof. Let $(\underline{\mathbf{u}}_h^n, \underline{p}_h^n)$ be the Stokes projection of $(\mathbf{u}^n, p^n)$ and $\underline{\sigma}_h^n$ be the interpolation of σ^n. We set

$$\mathbf{u}^n - \mathbf{u}_h^n = (\mathbf{u}^n - \underline{\mathbf{u}}_h^n) + (\underline{\mathbf{u}}_h^n - \mathbf{u}_h^n) =: \epsilon_{1h}^n + \mathbf{e}_{1h}^n$$

$$\sigma^n - \sigma_h^n = (\sigma^n - \underline{\sigma}_h^n) + (\underline{\sigma}_h^n - \sigma_h^n) =: \epsilon_{3h}^n + e_{3h}^n$$

and

$$p^n - p_h^n = (p^n - \underline{p}_h^n) + (\underline{p}_h^n - p_h^n) =: \epsilon_{2h}^n + e_{2h}^n\,.$$

Obviously, from the definition of π_h and (6), we have

$$\|\epsilon_{1h}^n\| + \|\epsilon_{2h}^n\| \leq C\, h^k\,, \quad \|\epsilon_{3h}^n\| \leq C\, h^{q+1} \quad \text{and} \quad \|\epsilon_{1h}^n\|_1 \leq C\, h^k\,.$$

Below, we need to only estimate $\mathbf{e}_{1h}^n$ and e_{3h}^n in the required norms. Subtracting (1) from (4) and using the definition of Stokes projection, we have

(10)
$$\begin{cases} (\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{v}_h) + a(\mathbf{e}_{1h}^n, \mathbf{v}_h) + b(\mathbf{v}_h, e_{2h}^n) &= \langle \mathcal{E}_h^n, \mathbf{v}_h \rangle \\ &+ (\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}_h\,, \\ b(\mathbf{e}_{1h}^n, r_h) &= 0 \qquad \forall r_h \in Q_h\,, \\ (\mathcal{D}e_{3h}^n, \tau_h) = (\partial_t \sigma^n - \mathcal{D}\underline{\sigma}_h^n, \tau_h) &+ \left\langle \widetilde{\mathcal{E}}_h^n, \tau_h \right\rangle + \left\langle \widehat{\mathcal{E}}_h^n, \tau_h \right\rangle \quad \forall \tau_h \in S_h \end{cases}$$

at each time step t_n, where $\mathcal{E}_h^n$, $\widetilde{\mathcal{E}}_h^n$ and $\widehat{\mathcal{E}}_h^n$ are defined by

$$\langle \mathcal{E}_h^n, \mathbf{v}_h \rangle \;:=\; c(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h) - c(\mathcal{I}(\mathbf{u}_h^n), \mathbf{u}_h^n, \mathbf{v}_h) + \tfrac{1}{Re}(\sigma^n - \mathcal{I}(\sigma_h^n), \mathbb{D}(\mathbf{v}_h))\,,$$

$$\left\langle \widetilde{\mathcal{E}}_h^n, \tau_h) \right\rangle \;:=\; B(\mathbf{u}^n, \sigma^n, \tau_h) - B(\mathcal{I}(\mathbf{u}_h^n), \sigma_h^n, \tau_h)\,,$$

and

$$\begin{aligned} \left\langle \widehat{\mathcal{E}}_h^n, \tau_h \right\rangle \;:=\;& (g_a(\sigma^n, \nabla \mathbf{u}^n), \tau_h) - (g_a(\mathcal{I}(\sigma_h^n), \nabla \mathbf{u}_h^n), \tau_h) \\ +\;& \tfrac{2\alpha}{\lambda}(\mathbb{D}(\mathbf{u}_h^n) - \mathbb{D}(\mathbf{u}^n), \tau_h) + \tfrac{1}{\lambda}(\sigma^n - \sigma_h^n, \tau_h)\,. \end{aligned}$$

We further decompose the error terms $\mathcal{E}_h^n$, $\widehat{\mathcal{E}}_h^n$ and $\widetilde{\mathcal{E}}_h^n$ as follows:

(11)
$$\begin{cases} \langle \mathcal{E}_h^n,\;\; \mathbf{v}_h \rangle = c(\mathbf{u}^n, \mathbf{u}^n - \underline{\mathbf{u}}_h^n, \mathbf{v}_h) + c(\mathbf{u}^n - \mathcal{I}(\mathbf{u}^n), \underline{\mathbf{u}}_h^n, \mathbf{v}_h) \\ \quad +\;\; c(\mathcal{I}(\mathbf{u}^n) - \mathcal{I}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{u}}_h^n, \mathbf{v}_h) - c(\mathcal{I}(\mathbf{e}_{1h}^n), \underline{\mathbf{u}}_h^n, \mathbf{v}_h) \\ \quad -\;\; c(\mathcal{I}(\underline{\mathbf{u}}_h^n), \mathbf{e}_{1h}^n, \mathbf{v}_h) - c(\mathcal{I}(\mathbf{e}_{1h}^n), \mathbf{e}_{1h}^n, \mathbf{v}_h) \\ \quad +\;\; \frac{1}{Re}(\sigma^n - \mathcal{I}(\sigma_h^n), \mathbb{D}(\mathbf{v}_h)) =: \displaystyle\sum_{i=1}^{7} \langle \mathcal{E}_i^n, \mathbf{v}_h \rangle\,, \end{cases}$$

(12)
$$\begin{cases} \left\langle \widetilde{\mathcal{E}}_h^n, \psi_h \right\rangle &= B(\mathcal{I}(\mathbf{u}_h^n), \sigma^n - \underline{\sigma}_h^n, \tau_h) \\ &+ B(\mathbf{u}^n - \mathcal{I}(\mathbf{u}^n), \sigma^n, \tau_h) + B(\mathcal{I}(\mathbf{u}^n) - \mathcal{I}(\underline{\mathbf{u}}_h^n), \sigma^n, \tau_h) \\ &- B(\mathcal{I}(\mathbf{e}_{1h}^n), \sigma^n, \tau_h) - B(\mathcal{I}(\mathbf{u}_h^n), e_{3h}^n, \tau_h) =: \displaystyle\sum_{i=1}^{5} \left\langle \widetilde{\mathcal{E}}_i^n, \tau_h \right\rangle\,. \end{cases}$$

and

(13)
$$\begin{cases} \left\langle \widehat{\mathcal{E}}_h^n, \tau_h \right\rangle &= (g_a(\sigma^n - \mathcal{I}(\sigma^n), \nabla \mathbf{u}^n), \tau_h) \\ &+ (g_a(\mathcal{I}(\sigma^n) - \mathcal{I}(\sigma_h^n), \nabla \mathbf{u}^n), \tau_h) \\ &+ (g_a(\mathcal{I}(\sigma_h^n), \nabla(\mathbf{u}^n - \mathbf{u}_h^n)), \tau_h) + \frac{2\alpha}{\lambda}(\mathbb{D}(\mathbf{u}_h^n) - \mathbb{D}(\mathbf{u}^n), \tau_h) \\ &+ \frac{1}{\lambda}(\sigma^n - \underline{\sigma}^n, \tau_h) - \frac{1}{\lambda}(e_{3h}^n, \tau_h) =: \displaystyle\sum_{i=1}^{6} \left\langle \widehat{\mathcal{E}}_i^n, \tau_h \right\rangle\,. \end{cases}$$

Now setting $(\mathbf{v}_h, \tau_h) = (\mathbf{e}_{1h}^n, e_{3h}^n)$ in (10) and using the skew symmetry of the trilinear form $c(\cdot,\cdot,\cdot)$ and coercivity of $B(\cdot,\cdot,\cdot)$, we obtain
(14)

$$\begin{cases} (\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{e}_{1h}^n) + \frac{2(1-\alpha)}{Re}\|\mathbb{D}(\mathbf{e}_{1h}^n)\|^2 = (\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{e}_{1h}^n) + \sum_{\substack{i=1 \\ i\neq 5, i\neq 6}}^{7} \langle \mathcal{E}_i^n, \mathbf{e}_{1h}^n\rangle \\ (\mathcal{D}e_{3h}^n, e_{3h}^n) + \frac{1}{\lambda}\|e_{3h}^n\|^2 + \frac{1}{2}\langle\langle e_{3h}^{n,+} - e_{3h}^{n,-}\rangle\rangle_{h,e_{3h}^n}^2 = (\partial_t \sigma^n - \mathcal{D}\underline{\sigma}_h^n, e_{3h}^n) \\ \qquad + \sum_{i=1}^5 \left\langle \widehat{\mathcal{E}}_i^n, e_{3h}^n\right\rangle + \sum_{i=1}^4 \left\langle \widetilde{\mathcal{E}}_i^n, e_{3h}^n\right\rangle . \end{cases}$$

We proceed to bound each term on the right-hand side of (14) and absorb like-terms into the left-hand side. The first terms on the right-hand side of (14) can be estimated with the help of (6) and (8). By Cauchy-Schwarz and triangle inequality, we have

$$(\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{e}_{1h}^n) \le \{\|\partial_t \mathbf{u}^n - \mathcal{D}\mathbf{u}^n\| + \|\mathcal{D}\mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n\|\}\,\|\mathbf{e}_{1h}^n\| .$$

Also notice since

$$\mathcal{D}\phi^n = \frac{3}{2\Delta t}\int_{t_{n-1}}^{t_n} \partial_t\phi\, dt - \frac{1}{2\Delta t}\int_{t_{n-2}}^{t_{n-1}} \partial_t\phi\, dt$$

we have

$$\|\mathcal{D}\phi^n\| \le \frac{C}{\sqrt{\Delta t}}\|\partial_t\phi\|_{L^2(t_{n-2},t_n;\mathbf{L}^2(\Omega))} .$$

Therefore by (6) and (8), we have
(15)

$$\begin{aligned} (\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}^n, \mathbf{e}_{1h}^n) \quad &\le \quad C\left\{(\Delta t)^{3/2}\|\partial_t^3 \mathbf{u}\|_{L^2(t_{n-2},t_n;\mathbf{L}^2(\Omega))}\right. \\ &+ \quad \left.\frac{h^k}{\sqrt{\Delta t}}\|(\partial_t \mathbf{u}, \partial_t p)\|_{L^2(t_{n-2},t_n;\mathbf{H}^{k+1}(\Omega)\times\mathbf{H}^k(\Omega))}\right\}\|\mathbf{e}_{1h}^n\| . \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} (\partial_t \sigma^n - \mathcal{D}\underline{\sigma}^n, e_{3h}^n) \quad &\le \quad C\left\{(\Delta t)^{3/2}\|\partial_t^3 \sigma\|_{L^2(t_{n-2},t_n;\mathbf{L}^2(\Omega))}\right. \\ &+ \quad \left.\frac{h^q}{\sqrt{\Delta t}}\|\partial_t \sigma\|_{L^2(t_{n-2},t_n;\mathbf{H}^{k+1}(\Omega))}\right\}\|e_{3h}^n\| . \end{aligned} \tag{16}$$

We now estimate the terms in $\left\langle \sum_{\substack{i=1 \\ i\neq 5, i\neq 6}}^{7} \mathcal{E}_i^n, \mathbf{e}_{1h}^n\right\rangle$, $\left\langle \sum_{i=1}^{5} \widehat{\mathcal{E}}_i^n, e_{3h}^n\right\rangle$ and $\left\langle \sum_{i=1}^{4} \widetilde{\mathcal{E}}_i^n, e_{3h}^n\right\rangle$. Using Hölders inequality, Gagliardo-Nirenberg's inequality, (6)-(7) and (9), we obtain

$$|\langle \mathcal{E}_1^n, \mathbf{e}_{1h}^n\rangle| \le C\|\mathbf{u}^n\|_1\|\mathbf{u}^n - \underline{\mathbf{u}}_h^n\|_1\|\mathbf{e}_{1h}^n\|_1 \le Ch^k\|\mathbf{u}\|_{C([t_{n-2},t_n];\mathbf{H}^k(\Omega))}\|\mathbf{e}_{1h}^n\|_1 ,$$

$$\begin{aligned} |\langle \mathcal{E}_2, \mathbf{e}_{1h}^n\rangle| \quad &\le \quad C\|\mathbf{u}^n - \mathcal{I}(\mathbf{u}^n)\|(\|\nabla\underline{\mathbf{u}}_h^n\|_{L^3(\Omega)} + \|\underline{\mathbf{u}}_h^n\|_\infty)\|\mathbf{e}_{1h}^n\|_1 \\ &\le \quad C(\Delta t)^{3/2}\|\partial_t^2 \mathbf{u}\|_{L^2(t_{n-2},t_n;\mathbf{L}^2(\Omega))}\|\mathbf{e}_{1h}^n\|_1 , \end{aligned}$$

$$\begin{aligned} |\langle \mathcal{E}_3^n, \mathbf{e}_{1h}^n\rangle| \quad &\le \quad C\|\mathcal{I}(\mathbf{u}^n) - \mathcal{I}(\underline{\mathbf{u}}_h^n)\|_1(\|\underline{\mathbf{u}}_h^n\|_\infty + \|\nabla\underline{\mathbf{u}}_h^n\|_{L^3(\Omega)})\|\mathbf{e}_{1h}^n\|_1 \\ &\le \quad Ch^k\|\mathbf{u}\|_{C([t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega))}\|\mathbf{e}_{1h}^n\|_1 , \end{aligned}$$

$$|\langle \mathcal{E}_4^n, \mathbf{e}_{1h}^n\rangle| \leq C\|\mathcal{I}(\mathbf{e}_{1h}^n)\|(\|\underline{\mathbf{u}}_h^n\|_\infty + \|\nabla\underline{\mathbf{u}}_h^n\|_{L^3(\Omega)})\|\mathbf{e}_{1h}^n\|_1$$

and

$$\begin{aligned}|\langle \mathcal{E}_7^n, \mathbf{e}_{1h}^n\rangle| &\leq C\left\{\|\sigma^n - \mathcal{I}(\sigma^n)\| + \|\mathcal{I}(e_{3h}^n)\|\right\}\|\mathbb{D}(\mathbf{e}_{1h}^n)\| \\ &\leq C\left\{(\Delta t)^{3/2}\|\partial_t^2\sigma\|_{L^2(t_{n-2},t_n;L^2(\Omega))} + \|\mathcal{I}(e_{3h}^n)\|\right\}\|\mathbb{D}(\mathbf{e}_{1h}^n)\|\,.\end{aligned}$$

Collecting these estimates, we obtain

$$\begin{aligned}&\left|\left\langle \sum_{\substack{i=1\\ i\neq5,i\neq6}}^{7} \mathcal{E}_i^n, \mathbf{e}_{1h}^n\right\rangle\right| \\ &\leq C\left\{h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n}(\|\mathbf{e}_{1h}^{i-1}\| + \|e_{3h}^{i-1}\|)\right\}\|\mathbf{e}_{1h}^n\|_1\,.\end{aligned}$$

We next estimate the terms in $\sum_{i=1}^4\left\langle \widehat{\mathcal{E}}_i^n, e_{4h}^n\right\rangle$ by Hölder's inequality, induction hypothesis (A2), (6) and (9). We obtain

$$\begin{aligned}|\left\langle \widehat{\mathcal{E}}_1^n, e_{3h}^n\right\rangle| &\leq C\|\nabla\mathbf{u}^n\|_\infty\|\sigma^n - \mathcal{I}(\sigma^n)\|e_{3h}^n\| \\ &\leq C(\Delta t)^{3/2}\|\partial_t^2\sigma\|_{L^2(t_{n-2},t_n;L^2(\Omega))}|\|e_{3h}^n\|\,,\end{aligned}$$

$$\begin{aligned}|\left\langle \widehat{\mathcal{E}}_2^n, e_{3h}^n\right\rangle| &\leq C\|\nabla\mathbf{u}^n\|_\infty\|\mathcal{I}(\sigma^n - \sigma_h^n)\|\|e_{3h}^n\| \\ &\leq C(\|\mathcal{I}(\sigma^n - \underline{\sigma}_h^n)\| + \|\mathcal{I}(\underline{\sigma}_h^n - \sigma_h^n)\|)\|e_{3h}^n\| \\ &\leq C(h^{q+1}\|\sigma\|_{C([t_{n-2},t_n];H^{q+1}(\Omega))} + \|\mathcal{I}(e_{3h}^n)\|)\|e_{3h}^n\|\,,\end{aligned}$$

$$\begin{aligned}|\left\langle \widehat{\mathcal{E}}_3^n, e_{3h}^n\right\rangle| &\leq C\|\mathcal{I}(\sigma_h^n)\|_\infty\|\nabla(\mathbf{u}^n - \underline{\mathbf{u}}_h^n)\| + \|\nabla(\underline{\mathbf{u}}_h^n - \mathbf{u}_h^n)\|\|e_{3h}^n\| \\ &\leq C(h^k\|\mathbf{u}\|_{C([t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega))} + \|\mathbb{D}(\mathbf{e}_{1h}^n)\|)\|e_{3h}^n\|\,,\end{aligned}$$

$$\begin{aligned}|\left\langle \widehat{\mathcal{E}}_4^n, e_{3h}^n\right\rangle| &\leq C(\|\mathbf{u}^n - \underline{\mathbf{u}}_h^n\|_1 + \|\mathbb{D}(\underline{\mathbf{u}}_h^n - \mathbf{u}_h^n)\|)\|e_{3h}^n\| \\ &\leq C(h^k\|\mathbf{u}\|_{C([t_{n-1},t_n];\mathbf{H}^{k+1}(\Omega))} + \|\mathbb{D}(\mathbf{e}_{1h}^n)\|)\|e_{3h}^n\|\end{aligned}$$

and

$$|\left\langle \widehat{\mathcal{E}}_5^n, e_{3h}^n\right\rangle| \leq C\|\sigma^n - \underline{\sigma}_h^n\| \leq C\,h^{q+1}\|\sigma\|_{C([t_{n-1},t_n];H^{q+1}(\Omega))}\,.$$

Collecting these estimates, we obtain

(17)
$$\sum_{i=1}^{5}\left\langle \widehat{\mathcal{E}}_i^n, e_{3h}^n\right\rangle \leq C\left\{(\Delta t)^{3/2} + h^{q+1} + h^k + \|\mathbb{D}(\mathbf{e}_{1h}^n)\| + \sum_{i=n-1}^{n}\|e_{3h}^{i-1}\|\right\}\|e_{3h}^n\|\,.$$

Let us next estimate the terms in $\sum_{i=1}^5\left\langle \widetilde{\mathcal{E}}_i^n, e_{3h}^n\right\rangle$. First notice that by the continuity of $\sigma^n - \underline{\sigma}_h^n$, we have that

$$\left\langle \widetilde{\mathcal{E}}_1^n, e_{3h}^n\right\rangle = (\mathcal{I}(\mathbf{u}_h^n)\cdot\nabla(\sigma^n - \underline{\sigma}_h^n), e_{3h}^n) + \frac{1}{2}(\nabla\cdot\mathcal{I}(\mathbf{u}_h^n)(\sigma^n - \underline{\sigma}_h^n), e_{3h}^n)\,.$$

Therefore we estimate it as follows

$$|\left\langle \widetilde{\mathcal{E}}_1^n, e_{3h}^n\right\rangle| \leq \left\{\|\mathcal{I}(\mathbf{u}_h^n)\|_\infty\|\nabla(\sigma^n - \underline{\sigma}_h^n)\| + \frac{1}{2}\|\nabla\mathcal{I}(\mathbf{u}_h^n)\|_\infty\|\sigma^n - \underline{\sigma}_h^n\|\right\}\|e_{3h}^n\|\,.$$

Employing the inverse inequality (3), induction hypothesis (A2) and the approximation property, we obtain

$$
\begin{aligned}
|\left\langle \widetilde{\mathcal{E}}_1^n, e_{3h}^n \right\rangle| &\leq \{\|\mathcal{I}(\mathbf{u}_h^n)\|_\infty \|\nabla(\sigma^n - \underline{\sigma}_h^n)\| \\
&+ Ch^{-1}\|\mathcal{I}(\mathbf{u}_h^n)\|_\infty \|\sigma^n - \underline{\sigma}_h^n\|\} \, \|e_{3h}^n\| \\
&\leq Ch^q \|\sigma\|_{C([t_{n-1},t_n];H^{q+1}(\Omega))} \|e_{3h}^n\| \, .
\end{aligned}
$$

For the term $\left\langle \widetilde{\mathcal{E}}_2^n, e_{3h}^n \right\rangle$, using the divergence free property of $\mathbf{u}^n$ and the continuity of σ, we can write it as $\left\langle \widetilde{\mathcal{E}}_2^n, e_{3h}^n \right\rangle = (\mathbf{u}^n - \mathcal{I}(\mathbf{u}^n) \cdot \nabla\sigma^n, e_{3h}^n)$. Thus estimating as usual, we obtain

$$
\begin{aligned}
|\left\langle \widetilde{\mathcal{E}}_2^n, e_{3h}^n \right\rangle| &\leq C\|\nabla\sigma^n\|_\infty \|\mathbf{u}^n - \mathcal{I}(\mathbf{u}^n)\| \|e_{3h}^n\| \\
&\leq C(\Delta t)^{3/2} \|\partial_t^2 \mathbf{u}\|_{L^2(t_{n-2},t_n;L^2(\Omega))} \|e_{3h}^n\| \, .
\end{aligned}
$$

Similarly for the terms $\left\langle \widetilde{\mathcal{E}}_3^n, e_{3h}^n \right\rangle$ and $\left\langle \widetilde{\mathcal{E}}_4^n, e_{3h}^n \right\rangle$, by the continuity of σ, we can estimate them as

$$
\begin{aligned}
|\left\langle \widetilde{\mathcal{E}}_3^n, e_{3h}^n \right\rangle| &\leq C\left\{\|\nabla\sigma^n\|_\infty \|\mathcal{I}(\mathbf{u}^n - \underline{\mathbf{u}}_h^n)\| + \tfrac{1}{2}\|\sigma^n\|_\infty \|\mathcal{I}(\mathbf{u}^n - \underline{\mathbf{u}}_h^n)\|_1\right\} \|e_{3h}^n\| \\
&\leq C\, h^k \|\mathbf{u}\|_{C([t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega))} \|e_{3h}^n\|
\end{aligned}
$$

and

$$
|\left\langle \widetilde{\mathcal{E}}_4^n, e_{3h}^n \right\rangle| \leq C\left\{\|\nabla\sigma^n\|_\infty \|\mathcal{I}(\mathbf{e}_{1h}^n)\| + \tfrac{1}{2}\|\sigma^n\|_\infty \|\nabla\mathcal{I}(\mathbf{e}_{1h}^n)\|_1\right\} \|e_{3h}^n\| \, .
$$

Collecting these estimates, we obtain

$$
\begin{aligned}
&\left|\left\langle \sum_{i=1}^4 \widetilde{\mathcal{E}}_i^n, e_{3h}^n \right\rangle\right| \\
&\leq C\left\{h^k + h^q + (\Delta t)^{3/2} + \sum_{i=n-1}^{n} (\|\mathbf{e}_{1h}^{i-1}\| + \|\mathbb{D}(\mathbf{e}_{1h}^{i-1})\|)\right\} \|e_{3h}^n\| \, .
\end{aligned}
$$

Employing the preceding estimates and (15)-(17) in (14) and using Young's inequality, we obtain

(18)

$$
\left\{
\begin{aligned}
(\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{e}_{1h}^n) + \tfrac{(1-\alpha)}{Re}\|\mathbb{D}(\mathbf{e}_{1h}^n)\|^2 &\leq C(\sum_{i=n-1}^{n} \|\mathbf{e}_{1h}^{i-1} + \|e_{3h}^{i-1}\|^2) + \Upsilon_1^n \\
(\mathcal{D}e_{3h}^n, e_{3h}^n) + \tfrac{1}{2\lambda}\|e_{3h}^n\|^2 &+ \tfrac{1}{2}\langle\langle e_{3h}^{n,+} - e_{3h}^{n,-}\rangle\rangle_{h,e_{3h}^n}^2 \leq \Upsilon_2^n \\
&+ C[\sum_{i=n-1}^{n+1} \|e_{3h}^{i-1}\|^2 + \sum_{i=n-1}^{n} \|\mathbf{e}_{1h}^{i-1}\|^2] \\
&+ \tfrac{(1-\alpha)}{4Re} \sum_{i=n-1}^{n} \|\mathbb{D}(\mathbf{e}_{1h}^{i-1})\|^2 \, .
\end{aligned}
\right.
$$

where

$$\Upsilon_1^n := C_1 \left\{ (\Delta t)^3 + \frac{h^{2k}}{\Delta t} + h^{2k} \right\} \text{ and } \Upsilon_2^n := C_2 \left\{ (\Delta t)^3 + \frac{h^{2k}}{\Delta t} + h^{2k} + h^{2q} \right\}.$$

Notice that by the regularity assumptions on the solution $(\mathbf{u}, p, \sigma)$, we have that

$$\Delta t \sum_{n=1}^{N} (\Upsilon_1^n + \Upsilon_2^n) \le C((\Delta t)^4 + h^{2k} + h^{2q}). \tag{19}$$

Thus adding $(18)_1 - (18)_2$ and summing the result from $n = 2$ to m, we obtain

(20)

$$\begin{aligned} \Delta t \sum_{n=2}^{m} ((\mathcal{D}\mathbf{e}_{1h}^n, \mathcal{D}e_{3h}^n), (\mathbf{e}_{1h}^n, e_{3h}^n)) &+ \tfrac{(1-\alpha)}{2Re} \Delta t \sum_{n=2}^{m} \|\mathbb{D}\mathbf{e}_{1h}^n\|^2 + \frac{\Delta t}{2\lambda} \sum_{n=2}^{m} \|e_{3h}^n\|^2 \\ &+ \tfrac{\Delta t}{2} \sum_{n=2}^{m} \langle\langle e_{3h}^{n,+} - e_{3h}^{n,-} \rangle\rangle^2 \\ &\le C((\Delta t)^4 + h^{2k} + h^{2q}) \\ &+ C\Delta t \sum_{n=2}^{m} (\|\mathbf{e}_{1h}^n\|^2 + \|e_{3h}^n\|^2). \end{aligned}$$

Finally notice that the BDF2 operator $\mathcal{D}$ satisfies the identity [19]

(21)

$$\begin{aligned} \sum_{n=2}^{m} \Delta t (\mathcal{D}(\phi^n), \phi^n) &= \tfrac{1}{4}\|\phi^m\|^2 + \tfrac{1}{4}\|2\phi^m - \phi^{m-1}\|^2 - \tfrac{1}{4}\|\phi^1\|^2 \\ &- \tfrac{1}{4}\|2\phi^1 - \phi^0\|^2 + \tfrac{1}{4} \sum_{n=2}^{m} \|\phi^n - 2\phi^{n-1} + \phi^{n-2}\|^2. \end{aligned}$$

Therefore, applying the discrete Grönwall lemma (Lemma 3.1) to (20), we obtain that

$$\begin{aligned} & \|(\mathbf{e}_{1h}^m, e_{3h}^m)\|^2 \\ + & \tfrac{(1-\alpha)\Delta t}{2Re} \sum_{n=2}^{m} \|\mathbb{D}\mathbf{e}_{1h}^n\|^2 + \frac{\Delta t}{2\lambda} \sum_{n=2}^{m} \|e_{3h}^n\|^2 \\ \le & C((\Delta t)^4 + h^{2k} + h^{2q}). \end{aligned}$$

The required error estimate now follows from (6) and triangle inequality. ∎

Verification of induction hypothesis (A2) Assume (A2) is true for

$n = 1, 2, \ldots, m-1$. By interpolation properties, inverse estimates, and Theorem 3.2, we have

$$
\begin{aligned}
\|\mathbf{u}_h^m\|_\infty &\leq \|\mathbf{u}_h^m - \mathbf{u}^m\|_\infty + \|\mathbf{u}^m\|_\infty \\
&\leq \|\mathbf{u}_h^m - \underline{\mathbf{u}}_h^m\|_\infty + \|\underline{\mathbf{u}}_h^m - \mathbf{u}^m\|_\infty + \|\mathbf{u}^m\|_\infty \\
&\leq Ch^{-d/2}[\|\mathbf{u}_h^m - \underline{\mathbf{u}}_h^m\| + \|\underline{\mathbf{u}}_h^m - \mathbf{u}^m\|] + M \\
&\leq Ch^{-d/2}[(\Delta t)^2 + h^k + h^q] + M\,.
\end{aligned} \tag{22}
$$

Therefore, if we set k and q such that $k - d/2 \geq 0$, $q - d/2 \geq 0$ and Δt, h such that $\Delta t < h^{d/4}/C$ then (6) implies $\|\mathbf{u}_h^n\|_\infty \leq \widehat{M}$. Similarly, we can show $\|\sigma_h^n\|_\infty \leq \widehat{M}$.

Theorem 3.3 Under the assumptions in Theorem 3.2, the approximate pressure p_h in (4) satisfies

$$
\|p - p_h\|_{l^2(L^2(\Omega))} \leq \frac{C}{\sqrt{\Delta t}}(\Delta t^2 + h^k + h^q)\,,
$$

for some constant C independent of mesh size h and time step Δt.

Proof. By the discrete inf-sup condition, it follows from $(10)_1$ that

(23)

$$
\begin{aligned}
\|\mathbf{e}_{2h}^n\| &\leq \frac{1}{\beta} \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{b(\mathbf{v}_h, \mathbf{e}_{2h}^n)}{\|\mathbf{v}_h\|_1} \\
&\leq \frac{1}{\beta} \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{1}{\|\mathbf{v}_h\|_1} \left\{ (\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{v}_h) + \left\langle \sum_{i=1}^{7} \mathcal{E}_i^n, \mathbf{v}_h \right\rangle - (\mathcal{D}\mathbf{e}_{1h}^n, \mathbf{v}_h) \right. \\
&- \left. a(\mathbf{e}_{1h}^n, \mathbf{v}_h) \right\}\,,
\end{aligned}
$$

where $\mathcal{E}_i^n$, $i = 1, \ldots, 7$ are as defined in the proof of Theorem 3.2. We can estimate the first term on the right-hand-side of (23) as we did in (15) to obtain

$$
(\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathbf{v}_h) \leq C \left\{ (\Delta t)^{3/2} + \frac{h^k}{\sqrt{\Delta t}} \right\} \|\mathbf{v}_h\|\,. \tag{24}
$$

The terms in $\left\langle \sum_{\substack{i=1 \\ i\neq 5, i\neq 6}}^{7} \mathcal{E}_i^n, \mathbf{v}_h \right\rangle$ can also be estimated as in the proof of Theorem 3.2 to obtain

(25)

$$
\left| \left\langle \sum_{\substack{i=1 \\ i\neq 5, i\neq 6}}^{7} \mathcal{E}_i^n, \mathbf{v}_h \right\rangle \right| \leq C \left\{ h^k + (\Delta t)^{3/2} + \sum_{i=n-1}^{n} \|\mathbf{e}_{1h}^{i-1}\| + \|e_{3h}^{i-1}\| \right\} \|\mathbf{v}_h\|_1\,.
$$

Let us next estimate $\langle \mathcal{E}_5^n, \mathbf{v}_h \rangle$ and $\langle \mathcal{E}_6^n, \mathbf{v}_h \rangle$. First notice by Theorem 3.2 and inverse inequality, we have that

(26)

$$
\begin{aligned}
\|\mathbf{e}_{1h}^n\|_1 &\leq C \min\{\|\mathbf{e}_{1h}^n\|/h, \|\mathbf{e}_{1h}^n\|_1\} \\
&\leq C \min\{(\Delta t^2 + h^k + h^q)/h, (\Delta t^2 + h^k + h^q)/(\Delta t)\} \leq C\,.
\end{aligned}
$$

Therefore by Hölder's, Gagliardo-Nirenberg inequality and (26), we obtain

$$(27)\quad \begin{cases} |\langle \mathcal{E}_5^n, \mathbf{v}_h\rangle| & \leq \; C(\|\mathcal{I}(\underline{\mathbf{u}}_h^n)\|_{L^3(\Omega)} + \|\mathcal{I}(\underline{\mathbf{u}}_h^n)\|_{L^\infty(\Omega)})\|\mathbf{e}_{1h}^n\|_1\|\mathbf{v}_h\|_1 \,, \\ |\langle \mathcal{E}_6^n, \mathbf{v}_h\rangle| & \leq \; C\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1\|\mathbf{e}_{1h}^n\|_1\|\mathbf{v}_h\|_1 \leq C\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1\|\mathbf{v}_h\|_1 \,. \end{cases}$$

Therefore employing estimates (24), (26) and (27) in (23), and estimating the last two terms on the right-hand side of (23) by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\mathbf{e}_{2h}^n\| \;\leq\; C\Bigg\{ & h^k + \tfrac{h^k}{\sqrt{\Delta t}} + (\Delta t)^{3/2} + \sum_{i=0}^{2}\|\mathbf{e}_{1h}^{n-i}\|_1 \\ + \; & \textstyle\sum_{i=1}^2[\|e_{1h}^{n-i}\| + \|e_{3h}^{n-i}\|] + \|\mathcal{D}\mathbf{e}_{1h}^n\|\Bigg\} \,. \end{aligned}$$

The required error estimate now follows from the last inequality by using Theorem 3.2 and triangle inequality. ■

Let us next derive optimal error estimates of the time derivatives of the velocity and use it to improve the error estimate of the pressure.

Corollary 3.4 Suppose the assumptions of Theorem 3.2 hold and assume $\mathbf{u} \in H^2(0,T;\mathbf{H}^1(\Omega))$ and $\sigma \in H^2(0,T;H^1(\Omega))$. In addition, assume the initial conditions $\mathbf{u}_h^i\,, i = 0,1$ satisfy $\sum_{i=0}^{1}\|\mathbf{u}(t_i) - \mathbf{u}_h^i\|_1 \leq h^k$ and $b(\mathbf{u}_h^i, r_h) = 0\,,\quad \forall r_h \in Q_h$. Then for any $h \in (0, h_0]$ the approximate velocity $\mathbf{u}_h^n$ satisfies

$$\|\partial_t\mathbf{u} - \mathcal{D}\mathbf{u}_h\|_{l^2(L^2(\Omega))} \leq c(\Delta t^2 + h^k + h^q)\,.$$

Moreover, we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(H^1(\Omega))} \leq c(\Delta t^2 + h^k + h^q)\,,$$

for some constant c independent of the mesh size h and time step Δt.

Proof. Putting $\mathbf{v}_h = \mathcal{D}(\mathbf{e}_{1h}^n)$ into (10) yields

$$(28)\qquad \|\mathcal{D}\mathbf{e}_{1h}^n\|^2 + a(\mathbf{e}_{1h}^n, \mathcal{D}\mathbf{e}_{1h}^n) = (\partial_t\mathbf{u}(t_n) - \mathcal{D}\underline{\mathbf{u}}_h^n, \mathcal{D}\mathbf{e}_{1h}^n) + \langle \mathcal{E}_h^n, \mathcal{D}\mathbf{e}_{1h}^n\rangle \,.$$

Let us use the identity

$$\begin{aligned} (\mathcal{D}(\phi^n), \phi^n) =& \frac{1}{2}\mathcal{D}(\|\phi^n\|^2) + \frac{1}{2\Delta t}[\|\phi^n - \phi^{n-1}\|^2 - \|\phi^{n-1} - \phi^{n-2}\|^2] \\ &+ \frac{1}{4\Delta t}\|\phi^n - 2\phi^{n-1} + \phi^{n-2}\|^2 \end{aligned}$$

to rewrite the bilinear form $a(\cdot,\cdot)$ on the left-hand side of (28) and also split up the nonlinear term $\langle \mathcal{E}_h^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle$ on the right-hand side of (28) as we did in the proof of Theorem 3.2. We obtain

$$(29)\quad \begin{aligned} & \|\mathcal{D}(\mathbf{e}_{1h}^n)\|^2 + \tfrac{(1-\alpha)}{Re}\mathcal{D}(\|\mathbb{D}\mathbf{e}_{1h}^n\|^2) + \tfrac{(1-\alpha)}{Re\Delta t}[\|\mathbb{D}(\mathbf{e}_{1h}^n - \mathbf{e}_{1h}^{n-1})\|^2 \\ & -\|\mathbb{D}(\mathbf{e}_{1h}^{n-1} - \mathbf{e}_{1h}^{n-2})\|^2] + \tfrac{(1-\alpha)}{2Re\Delta t}\|\mathbb{D}(\mathbf{e}_{1h}^n - 2\mathbf{e}_{1h}^{n-1} + \mathbf{e}_{1h}^{n-2})\|^2 \\ = \;& (\partial_t\mathbf{u}(t_n) - \mathcal{D}(\underline{\mathbf{u}}_h^n), \mathcal{D}(\mathbf{e}_{1h}^n)) + \sum_{i=1}^{7}\langle \mathcal{E}_i^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle \,, \end{aligned}$$

where $\mathcal{E}_i^n$, $i = 1, 2, \ldots, 7$, are as defined in the proof of Theorem 3.2. The first term on the right-hand-side of (29) can be estimated as in the proof of Theorem 3.2 to obtain

$$\begin{aligned} & (\partial_t \mathbf{u}^n - \mathcal{D}\underline{\mathbf{u}}^n, \mathcal{D}(\mathbf{e}_{1h}^n)) \\ \leq\ & C\left\{(\Delta t)^{3/2}\|\partial_t^3 \mathbf{u}\|_{L^2(t_{n-2},t_n;\mathbf{L}^2(\Omega))}\right. \\ +\ & \left.\frac{h^k}{\sqrt{\Delta t}}\|(\partial_t \mathbf{u}, \partial_t p)\|_{L^2(t_{n-2},t_n;\mathbf{H}^{k+1}(\Omega)\times\mathbf{H}^k(\Omega))}\right\}\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,. \end{aligned}$$

We estimate $\sum_{i=1}^{7}\langle \mathcal{E}_i^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle$ as usual using Hölder's inequality, Gagliardo-Nirenberg inequality, (6) and (9). We obtain

$$\begin{aligned} |\langle \mathcal{E}_1^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| &\leq C(\|\mathbf{u}(t_n)\|_\infty + \|\nabla \mathbf{u}(t_n)\|_{L^3})\|\mathbf{u}(t_n) - \underline{\mathbf{u}}_h^n\|_1\|\mathcal{D}(\mathbf{e}_{1h}^n)\| \\ &\leq C\,h^k\|(\mathbf{u}, p)\|_{C[t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega)\times H^k(\Omega))}\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,, \end{aligned}$$

$$\begin{aligned} |\langle \mathcal{E}_2^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| &\leq C\|\mathbf{u}(t_n) - \mathcal{I}(\mathbf{u}(t_n))\|_1(\|\underline{\mathbf{u}}_h^n\|_\infty + \|\nabla \underline{\mathbf{u}}_h^n\|_{L^3})\|\mathcal{D}(\mathbf{e}_{1h}^n)\| \\ &\leq C(\Delta t)^{3/2}\|\partial_t^2 \mathbf{u}\|_{L^2(t_{n-2},t_n;H^1(\Omega))}\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,, \end{aligned}$$

$$\begin{aligned} |\langle \mathcal{E}_3^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| &\leq C(\|\underline{\mathbf{u}}_h^n\|_\infty + \|\nabla \underline{\mathbf{u}}_h^n\|_{L^3})\|\mathcal{I}(\mathbf{u}^n - \underline{\mathbf{u}}_h^n)\|_1\|\mathcal{D}(\mathbf{e}_{1h}^n)\| \\ &\leq Ch^k\|(\mathbf{u}, p)\|_{C[t_{n-2},t_n];\mathbf{H}^{k+1}(\Omega)\times H^k(\Omega))}\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,, \end{aligned}$$

$$|\langle \mathcal{E}_4^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| \leq c(\|\underline{\mathbf{u}}_h^n\|_\infty + \|\nabla \underline{\mathbf{u}}_h^n\|_{L^3})\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,,$$

and

$$|\langle \mathcal{E}_5^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| \leq c(\|\mathcal{I}(\underline{\mathbf{u}}_h^n)\|_\infty + \|\nabla \mathcal{I}(\underline{\mathbf{u}}_h^n)\|_{L^3})\|\mathbf{e}_{1h}^n\|_1\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,.$$

From the inverse inequality (Assumption (A3)) and Sobolev inequality, it follows that

$$\|\phi_h\|_\infty + \|\nabla \phi_h\|_{L^3(\Omega)} \leq ch^{-\frac{d}{6}}\|\phi_h\|_1 \quad \forall \phi_h \in X^h\,. \tag{30}$$

Using (30), we can estimate $\langle \aleph_6^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle$ as follows

$$\begin{aligned} |\langle \mathcal{E}_6^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| &\leq [\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_\infty + \|\nabla \mathcal{I}(\mathbf{e}_{1h}^n)\|_{L^3}]\|\mathbf{e}_{1h}^n\|_1\|\mathcal{D}(\mathbf{e}_{1h}^n)\| \\ &\leq c^*\|\mathbf{e}_{1h}^n\|_1\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1 h^{-\frac{d}{6}}\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,. \end{aligned} \tag{31}$$

Alternatively, we can estimate $\langle \aleph_6^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle$ as follows

$$\begin{aligned} |\langle \mathcal{E}_6^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| &= |\tfrac{4}{\Delta t}c(\mathcal{I}(\mathbf{e}_{1h}^n), \mathbf{e}_{1h}^n, \mathbf{e}_{1h}^{n-1})| + |\tfrac{1}{4\Delta t}c(\mathcal{I}(\mathbf{e}_{1h}^n), \mathbf{e}_{1h}^n, \mathbf{e}_{1h}^{n-2})| \\ &\leq \tfrac{c^*}{\Delta t}\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1\|\mathbf{e}_{1h}^n\|_1[\|\mathbf{e}_{1h}^{n-1}\|_1 + \|\mathbf{e}_{1h}^{n-2}\|_1]\,. \end{aligned} \tag{32}$$

Combining (31) and (32), we have

$$|\langle \mathcal{E}_6^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| \leq c\gamma_n\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1[\|\mathcal{D}(\mathbf{e}_{1h}^n)\| + \sum_{i=1}^{2}\|\mathbf{e}_{1h}^{n-i}\|_1]\,, \tag{33}$$

where

$$\gamma_n = \min\{h^{-\frac{d}{6}}, (\Delta t)^{-1}\}\|\mathbf{e}_{1h}^n\|_1\,. \tag{34}$$

In order to estimate $\langle \mathcal{E}_7^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle$, we first write it as

$$\begin{aligned}\langle \mathcal{E}_7^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle &= -(\nabla\cdot(\sigma^n - \mathcal{I}(\sigma_h^n)), \mathcal{D}(\mathbf{e}_{1h}^n)) \\ &= -(\nabla\cdot(\sigma^n - \mathcal{I}(\sigma^n)), \mathcal{D}(\mathbf{e}_{1h}^n)) - (\nabla\cdot\mathcal{I}(\sigma^n - \underline{\sigma}_h^n), \mathcal{D}(\mathbf{e}_{1h}^n))\,.\end{aligned}$$

Then by Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle \mathcal{E}_7^n, \mathcal{D}(\mathbf{e}_{1h}^n)\rangle| &\le C[\|\sigma^n - \mathcal{I}(\sigma^n)\|_1 + \|\mathcal{I}(\sigma^n - \underline{\sigma}_h^n)\|_1 + \|\mathcal{I}(e_{3h}^n)\|]\|\mathcal{D}(\mathbf{e}_{1h}^n)\| \\ &\le C[(\Delta t)^{3/2}\|\partial_t^2\sigma\|_{L^2(t_{n-2},t_n;H^1(\Omega))} \\ &+ h^k\|\sigma\|_{C([t_{n-2},t_n];H^{k+1}(\Omega))} + \textstyle\sum_{i=n-1}^n \|e_{3h}^{i-1}\|]\|\mathcal{D}(\mathbf{e}_{1h}^n)\|\,.\end{aligned}$$

Employing these estimates in (29), we can write it as

$$\begin{aligned}&\tfrac{1}{2}\|\mathcal{D}(\mathbf{e}_{1h}^n)\|^2 + \tfrac{(1-\alpha)}{Re}\mathcal{D}(\|\mathbb{D}\mathbf{e}_{1h}^n\|^2) \\ &+ \tfrac{(1-\alpha)}{Re\Delta t}[\|\mathbb{D}(\mathbf{e}_{1h}^n - \mathbf{e}_{1h}^{n-1})\|^2 - \|\mathbb{D}(\mathbf{e}_{1h}^{n-1} - \mathbf{e}_{1h}^{n-2})\|^2] \\ &+ \tfrac{(1-\alpha)}{2Re\Delta t}\|\mathbb{D}(\mathbf{e}_{1h}^n - 2\mathbf{e}_{1h}^{n-1} + \mathbf{e}_{1h}^{n-2})\|^2 \\ &\le C\left\{\gamma_n^2\|\mathcal{I}(\mathbf{e}_{1h}^n)\|_1^2 + \alpha_n\right\},\end{aligned} \tag{35}$$

where

$$\begin{aligned}\alpha_n := &(\Delta t)^3\|\partial_t^3\mathbf{u}\|^2_{L^2(t_{n-2},t_n;L^2(\Omega))} \\ + &\tfrac{h^{2k}}{\Delta t}\|(\partial_t\mathbf{u}, \partial_t p)\|^2_{L^2(t_{n-2},t_n;\mathbf{H}^{k+1}\times H^k)} \\ + &h^{2k}\|(\mathbf{u},p)\|^2_{C([t_{n-2},t_n];\mathbf{H}^{k+1}\times H^k)} + h^{2k}\|\sigma\|^2_{C([t_{n-2},t_n];H^{k+1})} \\ + &(\Delta t)^3\|\partial_t^2\sigma\|^2_{L^2(t_{n-2},t_n;H^1(\Omega))} + (\Delta t)^3\|\partial_t^2\mathbf{u}\|^2_{L^2(t_{n-2},t_n;\mathbf{H}^1(\Omega))} \\ + &\sum_{i=0}^2\|\mathbf{e}_{1h}^{n-i}\|_1^2 + \sum_{i=1}^2\|e_{3h}^{n-i}\|_1^2\,.\end{aligned}$$

Notice by the telescoping property and the identity $\frac{3}{2}\|a\|^2 - \frac{1}{2}\|b\|^2 + \|a-b\|^2 = \frac{1}{2}\|a\|^2 + (\sqrt{2}a - \frac{1}{\sqrt{2}}b)^2$, we have

$$\begin{aligned}\sum_{n=2}^m\big\{\mathcal{D}(\|\phi^n\|^2) &+ \tfrac{1}{\Delta t}\left[\|\phi^n - \phi^{n-1}\|^2 - \|\phi^{n-1} - \phi^{n-2}\|^2\right]\big\} \\ &= \tfrac{1}{2\Delta t}\|\phi^m\|^2 + \tfrac{1}{2\Delta t}\|2\phi^m - \phi^{m-1}\|^2 - \tfrac{1}{2\Delta t}\|\phi^1\|^2 \\ &- \tfrac{1}{2\Delta t}\|2\phi^1 - \phi^0\|^2\,.\end{aligned} \tag{36}$$

Therefore summing from $n = 2$ to m and using the assumption about initial approximations, we obtain

(37)

$$\begin{aligned}\frac{\Delta t}{2}\sum_{i=2}^{m}\|\mathcal{D}(\mathbf{e}_{1h}^{n})\|^2 &+ \frac{(1-\alpha)}{2Re}\|\mathbb{D}(\mathbf{e}_{1h}^{m})\|^2 + \frac{(1-\alpha)}{2Re}\|2\mathbb{D}(\mathbf{e}_{1h}^{m}) - \mathbb{D}(\mathbf{e}_{1h}^{m-1})\|^2 \\ &+ \frac{(1-\alpha)}{2Re}\sum_{n=2}^{m}\|\mathbb{D}(\mathbf{e}_{1h}^{n} - 2\mathbf{e}_{1h}^{n-1} + \mathbf{e}_{1h}^{n-2})\|^2 \\ &\le C\left\{\sum_{n=2}^{m}\Delta t\gamma_n^2\|\mathcal{I}(\mathbf{e}_{1h}^{n})\|_1^2 + \sum_{i=2}^{m}\alpha_n\Delta t + h^{2k}\right\}.\end{aligned}$$

By the regularity assumptions on the solution $(\mathbf{u}, p, \sigma)$ and the error bounds of Theorem 3.2, we have

$$\Delta t\sum_{i=1}^{N}\alpha_i \le C((\Delta t)^4 + h^{2k} + h^{2q}). \tag{38}$$

Moreover, by (34) and the error bounds of Theorem 3.2, we have

$$\begin{aligned}\Delta t\sum_{i=1}^{N}\gamma_i^2 &\le \min\{h^{-\frac{d}{3}}, (\Delta t)^{-2}\}\Delta t\sum_{i=1}^{N}\|\mathbf{e}_{1h}^{i}\|_1^2 \\ &\le C\min\{h^{-\frac{d}{3}}, (\Delta t)^{-2}\}(h^{2k} + h^{2q} + (\Delta t)^4) \\ &\le C\min\{h^{2k-\frac{d}{3}} + h^{2q-\frac{d}{3}} + (\Delta t)^2\} \le C.\end{aligned}$$

Therefore by applying discrete Gronwall lemma to (37), we obtain

(39)

$$\begin{aligned}\Delta t\sum_{i=2}^{m}\|\mathcal{D}(\mathbf{e}_{1h}^{n})\|^2 &+ \frac{(1-\alpha)}{Re}\|\mathbb{D}(\mathbf{e}_{1h}^{m})\|^2 + \frac{(1-\alpha)}{Re}\|2\mathbb{D}(\mathbf{e}_{1h}^{m}) - \mathbb{D}(\mathbf{e}_{1h}^{m-1})\|^2 \\ &+ \frac{(1-\alpha)}{Re}\sum_{n=2}^{m}\|\mathbb{D}(\mathbf{e}_{1h}^{n} - 2\mathbf{e}_{1h}^{n-1} + \mathbf{e}_{1h}^{n-2})\|^2 \\ &\le C((\Delta t)^4 + h^{2k} + h^{2q}).\end{aligned}$$

It now follows from (39) and (6) that

$$\begin{aligned}\Delta t\sum_{n=2}^{m}\|\mathcal{D}(\mathbf{u}^n) - \mathcal{D}(\mathbf{u}_h^n)\|^2 &\le C((\Delta t)^4 + h^{2k} + h^{2q}) \\ \|\mathbf{u}^n - \mathbf{u}_h^n\|_1^2 &\le C((\Delta t)^4 + h^{2k} + h^{2q}).\end{aligned} \tag{40}$$

Finally $(40)_1$ and (8) yields

$$\|\partial_t\mathbf{u} - \mathcal{D}\mathbf{u}_h\|_{l^2(L^2(\Omega))} \le c(\Delta t^2 + h^k + h^q).$$

□

Corollary 3.5 Suppose the assumptions of Corollary 3.4 hold. Then the approximate pressure p_h^n in (4) satisfies

$$\|p - p_h\|_{l^2(L^2(\Omega))} \le C(\Delta t^2 + h^k + h^q).$$

Proof. A proof of this Corollary can be furnished by arguing along the same line as in the proof of Theorem 3.3. Therefore we only sketch it here. It follows from the error bounds in Corollary 3.4

$$\Delta t\|\mathcal{D}\mathbf{e}_{1h}^n\|^2 \le c((\Delta t)^4 + h^{2k} + h^{2q})\,. \tag{41}$$

Therefore using (41) in (27), we obtain the required estimate. □

4. Numerical results

In this section, we present two numerical examples to demonstrate the convergence and stability of the algorithm proposed in this paper. The spatial domain Ω is partitioned using a structured triangular mesh. The spatial discretization is effected via the (P_2, P_1, P_2dc) element, i.e., the continuous piecewise quadratics and continuous piecewise linear finite element spaces (Taylor-Hood finite element pair) for the fluid velocity and pressure approximations, and discontinuous piecewise quadratics for the stress tensor.

4.1. Example 1: Accuracy test. In this example, we report some numerical simulations to test the convergence theory presented in Theorem 3.2. Taking the spatial domain to be $\Omega = (0,1) \times (0,1)$, the time interval to be $[0,T] = [0,1]$ and the parameters to be $Re = 1.0, \lambda = 1.0, a = 0, \alpha = 0.5$, we consider the exact solution $(\mathbf{u}, p, \sigma)$ given by

$$\mathbf{u} = (-(1-\cos(2\pi x))\sin(2\pi y)e^{-t}, \sin(2\pi x)(1-\cos(2\pi y))e^{-t})\,,$$

$$\sigma = 2\alpha\mathbb{D} \quad \text{and} \quad p = (\sin(4\pi x) + \sin(4\pi y))e^{-t}\,.$$

satisfying the divergence free condition. The source terms (right-hand sides), initial conditions and boundary conditions are chosen to correspond the exact solution.

The performance of the numerical scheme studied herein is also compared with the coupled scheme (monolithic, fully implicit scheme) derived by setting $\mathcal{I}(\mathbf{u}_h^n) = \overline{\mathbf{u}}_h^n$ and $\mathcal{I}(\sigma_h^n) = \overline{\sigma}_h^n$ in Algorithm 2.1. The monolithic scheme requires a system of nonlinear algebraic equations to be solved using an iterative method at each time step. We employ Newton iterative method for solving this nonlinear algebraic equations and the iteration is stopped when relative nonlinear residual is less than 10^{-6}. For both the monolithically coupled scheme and the decoupled scheme, a banded Gaussian elimination is used to solve the linear algebraic systems.

First, we compare the errors with both the decoupled scheme and the monolithic coupled scheme. In Table 1, we compare both schemes at time $t_N = 1.0$, with varying spacing $h = 1/2, 1/4, 1/8, 1/16, 1/32$ and fixed time step$\Delta t = 0.01$. As can be seen from Table 1, the error estimates of $\mathbf{u}$ and σ in Theorem 3.2 for the orders of convergence in space agree with computational results.

Moreover, it can be seen that both the schemes achieve similar precision. In order to determine the order of convergence α with respect to the time

TABLE 1. Convergence performance of the decoupled and monolithic (coupled) schemes at time $t_N = 1.0$, with fixed time step $\Delta t = 0.01$.

h	Coupled Scheme		Decoupled Scheme	
	$\|\mathbf{u}^N - \mathbf{u}_h^N\|$	$\|\sigma^N - \sigma_h^N\|$	$\|\mathbf{u}^N - \mathbf{u}_h^N\|$	$\|\sigma^N - \sigma_h^N\|$
$\frac{1}{2}$	0.03414835	0.02563487	0.03425877	0.025748644
$\frac{1}{4}$	0.00858735	0.00657473	0.00883213	0.00664378
$\frac{1}{8}$	0.00215406	0.00166191	0.00226471	0.00170401
$\frac{1}{16}$	0.00054004	0.00041678	0.00057288	0.00042733
$\frac{1}{32}$	0.00013505	0.000093346	0.00013729	0.00010686

TABLE 2. Convergence order of $O(\Delta t^{\alpha})$ of the decoupled scheme at time $t_N = 1.0$, with the fixed spacing $h = \frac{1}{32}$.

Δt	$\|\mathbf{u}^N - \mathbf{u}_h^N\|$	Order	$\|\sigma^N - \sigma_h^N\|$	Order
1/20	4.2144355×10^{-5}	-	3.8766845×10^{-5}	-
1/40	$1.07092779 \times 10^{-5}$	1.9764782	$0.98340386 \times 10^{-5}$	1.9789674
1/80	$0.27015727 \times 10^{-5}$	1.9869898	$0.24761613 \times 10^{-5}$	1.9896787
1/160	$0.06780444 \times 10^{-5}$	1.9943478	$0.06204215 \times 10^{-5}$	1.9967848

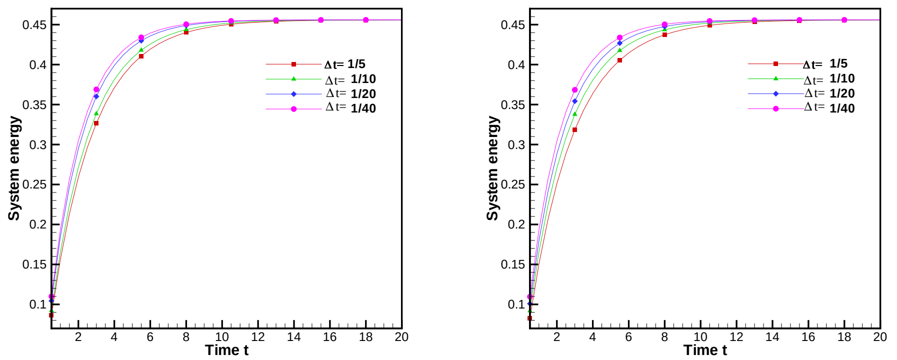
step Δt, we fix the spatial spacing h and use the following approximation

$$\alpha \approx \log_2 \frac{\|\mathbf{v}_{h,\Delta t}(x, t_N) - \mathbf{v}_{h,\frac{\Delta t}{2}}(x, t_N)\|}{\|\mathbf{v}_{h,\frac{\Delta t}{2}}(x, t_N) - \mathbf{v}_{h,\frac{\Delta t}{4}}(x, t_N)\|}. \tag{42}$$

In Table 2, we list the values of the right-hand side of (42) with a fixed spacing $h = 1/32$ and varying time step $\Delta t = 1/20, 1/40, 1/80, 1/160$. As can be seen the orders of convergence in time are all second order for the decoupled scheme suggesting that the orders of convergence in time in error estimates in Theorem 3.2 for the L^2- norm of $\mathbf{u}$, ϕ and σ are optimal.

4.2. Example 2: Stability test. In this example, we simulate viscoelastic flow through $4:1$ abrupt contraction with centerline symmetry, a prototypical problem for viscoelastic fluid flow. The computational domain is chosen to be $\Omega := (0, 10) \times (0, 1) \setminus (4, 10) \times (1/4, 1)$. It is assumed that the channel lengths are sufficiently long for fully developed Poiseuille flow at both the inflow and outflow boundaries,

$$\Gamma_{in} := \{(x, y) \ : \ x = 0, \ 0 \le y \le 1 \}$$



(a) (b)

FIGURE 1. System energy with different time steps: (a) System energy with decoupled BDF2 scheme, (b) System energy with coupled BDF2 scheme.

and

$$\Gamma_{out} := \{(x,y) \ : \ x = 10, \ 0 \leq y \leq 1/4 \} .$$

The parameters Re, α, λ and a are chosen to be $1, 8/9, 0.7$ and 1, respectively. We impose zero boundary condition for the velocity on the channel walls and specify symmetric natural boundary condition on the centerline ($y = 0,\ 0 \leq x \leq 10$) of the spatial domain. Fully developed flow conditions are applied for the fluid velocity and viscoelastic stress tensor at the inlet but only for the velocity at the outlet, i.e.,

$$\mathbf{u} = ((1-y^2)/32, 0) \quad \text{on} \quad \Gamma_{in} , \qquad \mathbf{u} = (2(1/16 - y^2), 0) \quad \text{on} \quad \Gamma_{out} ,$$

and

$$\sigma_{11} = 2\lambda y^2 \alpha/256 , \quad \sigma_{12} = \sigma_{21} = -\alpha y/16 \text{ and } \sigma_{22} = 0 \quad \text{on} \quad \Gamma_{in} .$$

The discrete system energy defined by $E_n := \|\mathbf{u}_h^n\|^2 + \|\sigma_h^n\|^2$ was computed with these data for varying time step size Δt and fixed $\Delta x = 0.001$. In Figure 1, we present the time evolution of the discrete system energy E_n for four time step sizes $\Delta t = 1/5, 1/10, 1/20, 1/40$ but for fixed spacing $h = 1/100$ until $T = 20$. We observe that all four energy curves show that steady state is reached for all time step sizes for both coupled and decoupled BDF2 schemes. Moreover, we observe that the proposed algorithm is stable with no restriction on Δt.

5. Concluding remarks

We proposed and investigated an accurate and efficient second-order decoupled time stepping scheme for solving viscoelastic fluid flow system. A second order extrapolation was used to effect a decoupling of the system so that two decoupled problems could be solved at each time step. A discontinuous Galerkin finite element method was used to spatially approximate the constitutive equation for the extra stress tensor while standard Galerkin finite element method was used to approximate the Navier-Stokes equations. A priori error estimates are derived for the fully discrete system assuming the mesh size Δx and time step step size Δt satisfy $\Delta t \leq C\, h^{d/4}$. Numerical results presented confirm the theoretical error and stability estimates.

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