INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 1, Pages 1–18

ADAPTIVE MULTIGRID METHOD FOR EIGENVALUE PROBLEM

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Abstract. In this paper, we propose a type of adaptive multigrid method for eigenvalue problem based on the multilevel correction method and adaptive multigrid method. Different from the standard adaptive finite element method applied to eigenvalue problem, with our method we only need to solve a linear boundary value problem on each adaptive space and then correct the approximate solution by solving a low dimensional eigenvalue problem. Further, the involved boundary value problems are solved by some adaptive multigrid iteration steps. The proposed adaptive algorithm can reach the same accuracy as the standard adaptive finite element method for eigenvalue problem but evidently reduces the computational work. In addition, the corresponding convergence and optimal complexity analysis are derived theoretically and numerically, respectively.

Key words. Eigenvalue problem, adaptive multigrid method, multilevel correction, convergence, optimal complexity.

1. Introduction

How to solve large-scale eigenvalue problems is a very significant problem in modern scientific and engineering calculations. Many physical models and engineering models ultimately boil down to eigenvalue problems, such as the structural vibration analysis in buildings design, stability analysis in control systems, inherent frequency analysis of aircraft, etc. In recent years, the first-principles electronic structure calculations have pushed into the spotlight, and its key point is right to solve a class of nonlinear eigenvalue models. Therefore, it is necessary to make an indepth study of eigenvalue problem for its important theoretical significance and wide application value.

Among different numerical methods for eigenvalue problems, the adaptive finite element method (AFEM) is an efficient approach in generating optimal triangulation. AFEM was proposed by Babuška and his cooperators in [4, 5]. Up to now, the corresponding theoretical analysis of AFEM is well-developed. The convergence and optimal complexity analysis for boundary value problem can be found in [10, 16, 22, 23, 34, 33, 35, 37]. For eigenvalue problems, we can also find some similar results in [15, 17, 18, 19, 27, 30]. To further improve the efficiency of adaptive finite element method, the multilevel technique was absorbed to generate the adaptive multigrid method. Actually, it is worthing noting that adaptive mesh refinement technique was confirmed fully compatible with the multilevel mesh structure. Based on this idea, Brandt [6, 8] introduced the multilevel adaptive technique (MLAT), and McCormick [31] developed the fast adaptive composite grid method (FAC). For more results about the adaptive multigrid method, please refer to [13, 21, 32, 38, 39] and the references cited therein.

Though the optimal triangulations can be derived by standard AFEM, we have to solve an eigenvalue problem on each adaptive space, which is time-consuming

Received by the editors January 13, 2019 and, in revised form, August 10, 2020.
2000 *Mathematics Subject Classification.* 65F15, 65N15, 65N25, 65N30, 65N50.
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and very tedious with the growth of degree of freedoms. The purpose of this paper is to propose a new type of adaptive multigrid method for solving eigenvalue problem based on adaptive finite element method, adaptive multigrid method and the recent work on the multilevel correction method [12, 24, 26, 28, 29, 40, 41]. In addition, we also analyze the corresponding convergence and optimal complexity property. In our presented adaptive multigrid method, the eigenvalue problem can be transformed into a series of linear boundary value problems on the fine grids and some eigenvalue problems on the coarsest grid. The dimension of the small-scale eigenvalue problem will be fixed during the adaptive refinement, thus the solving time can be ignored if the size of mesh becomes increasingly smaller after some refinement steps. Further, for the involved linear boundary value problems, we only need to proceed some multigrid iteration steps on the newly refined elements and their neighbors. For more details, please refer to [6, 21, 38, 39] and references cited therein. In this paper, we will adopt the techniques in [10, 15, 22] to prove the convergence and optimal complexity of the proposed adaptive multigrid method.

The rest of the paper is arranged as follows. Section 2 describes some basic notations and the standard AFEM for the second order elliptic boundary value problem. In Section 3, we introduce the adaptive multigrid method for eigenvalue problems. The corresponding convergence and complexity analysis will be given in Section 4. In Section 5, some numerical experiments are presented to validate the efficiency and the theoretical analysis. Section 6 concludes.

2. Preliminaries of standard adaptive finite element method for boundary value problem

This section is devoted to introducing some basic notation and some useful results of AFEM for second order linear elliptic boundary value problems. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see, e.g., [1]). For $p=2$, we denote $H^s(\Omega)=W^{s,2}(\Omega)$, $H_0^1(\Omega)=\{v\in H^1(\Omega):\ v|_{\partial\Omega}=0\}$, where $v|_{\partial\Omega}=0$ is in the sense of trace and $\|\cdot\|_{s,\Omega}=\|\cdot\|_{s,2,\Omega}$. For simplicity, we set $V=H_0^1(\Omega)$ in the rest of this paper.

Here, we consider the following homogeneous boundary value problem:

$$\left\{\begin{array}{rcll} Lu:=-\nabla\cdot(\mathcal{A}\nabla u)+\phi u &=& f & \text{in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega,\end{array}\right. \tag{1}$$

where $\mathcal{A}=(a_{i,j})_{d\times d}$ is a symmetric positive definite matrix with suitable regularity and ϕ is a nonnegative function.

In order to use the finite element method, we first introduce the weak form for (1) as follows: Find $u\in V$ such that

$$a(u,v)=(f,v),\quad \forall v\in V, \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v)=\int_\Omega(\mathcal{A}\nabla u\cdot\nabla v+\phi uv)d\Omega. \tag{3}$$

Obviously, the bilinear form $a(\cdot,\cdot)$ is bounded and coercive over V. Thus, we can define the energy norm $\|\cdot\|_{a,\Omega}$ by $\|w\|_{a,\Omega}=\sqrt{a(w,w)}$.

Now, we introduce the standard finite element method for linear boundary value problem (2). Firstly, we generate a shape regular decomposition of the computing domain $\Omega\subset\mathcal{R}^d$ $(d=2,3)$ into triangles or rectangles for $d=2$, tetrahedrons or hexahedrons for $d=3$ (cf. [9, 14]). Then we consider the finite element discretization on the shape regular family of nested conforming meshes $\{\mathcal{T}_k\}$ over Ω with the mesh diameter h_k.

Let $V_k \subset V$ be the corresponding family of nested linear finite element spaces of continuous piecewise polynomials over $\mathcal{T}_k$. Based on V_k, we can define the finite element scheme for linear boundary value problem (2): Find $u_k \in V_k$ such that

$$a(u_k, v_k) = (f, v_k), \quad \forall v_k \in V_k. \tag{4}$$

For the purpose of theoretical analysis, we define the Galerkin projection $P_k : H_0^1(\Omega) \to V_k$ by

$$a(u - P_k u, v_k) = 0, \quad \forall v_k \in V_k. \tag{5}$$

Then we can obtain $u_k = P_k u$ and

$$\|P_k u\|_{a,\Omega} \leq \|u\|_{a,\Omega}, \quad \forall u \in V. \tag{6}$$

Based on the conclusion of adaptive finite element method for boundary value problem (see, e.g. [10, 33, 34]), we propose the a posteriori error estimator for finite element problem (4). We define the element residual $\widehat{\mathcal{R}}_T(u_k)$ and the jump residual $\widehat{\mathcal{J}}_E(u_k)$ as follows

$$\begin{aligned}
\widehat{\mathcal{R}}_T(u_k) &:= f - Lu_k = f - \phi u_k + \nabla\cdot(\mathcal{A}\nabla u_k) \ \text{ in } T \in \mathcal{T}_k,\\
\widehat{\mathcal{J}}_E(u_k) &:= -\nabla\mathcal{A}u_k^+ \cdot \nu^+ - \nabla\mathcal{A}u_k^- \cdot \nu^- := [[\nabla\mathcal{A}u_k]]_E \cdot \nu_E \ \text{ on } E \in \mathcal{E}_k,
\end{aligned}$$

where E is the common side of elements T^+ and T^- with outward normals ν^+ and ν^-, $\nu_E = \nu^-$.

Then for any element $T \in \mathcal{T}_k$ with diameter h_T, the local error indicator and oscillation can be defined by

$$\begin{aligned}
\widehat{\eta}_k^2(u_k;T) &:= h_T^2\|\widehat{\mathcal{R}}_T(u_k)\|_{0,T}^2 + \sum_{E\in\mathcal{E}_k, E\subset\partial T} h_E\|\widehat{\mathcal{J}}_E(u_k)\|_{0,E}^2,\\
\widehat{\operatorname{osc}}_k^2(u_k;T) &:= h_T^2\|(I-P_T)\widehat{\mathcal{R}}_T(u_k)\|_{0,T}^2 + \sum_{E\in\mathcal{E}_k, E\subset\partial T} h_E\|(I-P_E)\widehat{\mathcal{J}}_E(u_k)\|_{0,E}^2,
\end{aligned}$$

where P_T and P_E are the L^2-projection operators to polynomials of some degree on T and E, respectively.

Given a submesh $\omega \subset \mathcal{T}_k$, the global error indicator and oscillation can be defined by

$$\widehat{\eta}_k^2(u_k;\omega) := \sum_{T\in\omega}\widehat{\eta}_k^2(u_k;T) \quad \text{and} \quad \widehat{\operatorname{osc}}_k^2(u_k;\omega) := \sum_{T\in\omega}\widehat{\operatorname{osc}}_k^2(u_k;T).$$

Now we recall the reliability and efficiency of the a posterior error estimator in the following lemma.

Lemma 2.1. *There exist two constants $\hat{C}_{\mathrm{u}}$ and $\hat{C}_\ell$, depending only on the shape-regularity of $\mathcal{T}_k$, such that the following reliability and efficiency hold*

$$\|u - u_k\|_{a,\Omega}^2 \leq \hat{C}_{\mathrm{u}}\widehat{\eta}_k^2(u_k;\mathcal{T}_k) \tag{7}$$

and

$$\hat{C}_\ell\widehat{\eta}_k^2(u_k;\mathcal{T}_k) \leq \|u-u_k\|_{a,\Omega}^2 + \widehat{\operatorname{osc}}_k^2(u_k;\mathcal{T}_k). \tag{8}$$

The standard adaptive finite element method can be written as loop of the form

Solve → Estimate → Mark → Refine.

More precisely, to get $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$, we first need to solve the finite element equation (4) on $\mathcal{T}_k$ to get the approximate solution and then calculate the local error indicator on each mesh element. Next we mark elements to be subdivided according to the local error indicator, and then refine these elements in such a way that the triangulation is still shape regular and conforming.

In order to further simplify the description of adaptive algorithm, we first introduce some modules for boundary value problem (1):

- $w_k = \text{BVP\_SOLVE}(f, V_k)$: Solve the linear boundary value problem (1) in the finite element space V_k and output the discrete solution w_k.
- $w_k = \text{MGBVP\_SOLVE}(f, w_0, V_k)$: Solve the linear boundary value problem (1) by multigrid method with initial value w_0 in the finite element space V_k and output the iteration solution w_k.
- $\{\widehat{\eta}_k(u_k;T)\}_{T\in\mathcal{T}_k} = \text{BVP\_ESTIMATE}(u_k, \mathcal{T}_k)$: Compute the local error indicator on each element.
- $\mathcal{M}_k = \text{BVP\_MARK}(\theta, \widehat{\eta}_k(u_k;T), \mathcal{T}_k)$: Construct a subset $\mathcal{M}_k$ by Dörfler's marking strategy presented in [16], i.e., construct a minimal subset $\mathcal{M}_k$ from $\mathcal{T}_k$ by filtrating relevant elements in $\mathcal{T}_k$ such that

$$\widehat{\eta}_k(u_k;\mathcal{M}_k) \geq \theta \widehat{\eta}_k(u_k;\mathcal{T}_k).$$

- $(\mathcal{T}_{k+1}, V_{k+1}) = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$: Output a conforming refinement $\mathcal{T}_{k+1}$ according to $\mathcal{M}_k$ where at least all element of $\mathcal{M}_k$ are refined and construct the corresponding finite element space V_{k+1}.

Then we present the standard AFEM for boundary value problem (1) as follows:

Adaptive Algorithm 1

Given a parameter $\theta \in (0,1)$ and an initial mesh $\mathcal{T}_1$. Set $k := 1$ and do the following loops:

(1) $u_k = \text{BVP\_SOLVE}(f, V_k)$;
(2) $\{\widehat{\eta}_k(u_k;T)\}_{T\in\mathcal{T}_k} = \text{BVP\_ESTIMATE}(u_k, \mathcal{T}_k)$;
(3) $\mathcal{M}_k = \text{BVP\_MARK}(\theta, \widehat{\eta}_k(u_k;T), \mathcal{T}_k)$;
(4) $(\mathcal{T}_{k+1}, V_{k+1}) = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$;
(5) Set $k := k+1$ and go to step 1.

We now recall some well-known results of Adaptive Algorithm 1 for elliptic boundary value equations (see [10]), which will be used in the following analysis.

Theorem 2.1. *([10]) Let $\{u_k\}$ be a sequence of approximate solutions for (1) which are produced by Adaptive Algorithm 1. Then, there exist constants $\hat{\gamma} > 0$ and $\hat{\xi} \in (0,1)$, depending on the shape regularity of meshes and marking parameter θ, such that any two consecutive iterates k and $k+1$ have the following property*

$$\|u - u_{k+1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_{k+1}^2(u_{k+1};\mathcal{T}_{k+1}) \leq \hat{\xi}^2\big(\|u-u_k\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_k^2(u_k;\mathcal{T}_k)\big). \tag{9}$$

In this paper, we assume that the marking parameter θ satisfies $\theta \in (0,\theta_*)$ with θ_* being defined in Assumption 5.8 of [10].

Lemma 2.2. *([10]) Let $\mathcal{T}_{k,*}$ be a refinement of $\mathcal{T}_k$. Suppose the projections $P_{k,*}u$ and $P_k u$ satisfy the following decrease property*

$$\|u - P_{k,*}u\|_{a,\Omega}^2 + \widehat{\text{osc}}_{k,*}^2(P_{k,*}u;\mathcal{T}_{k,*}) \leq \tilde{\xi}_0^2\big(\|u-P_k u\|_{a,\Omega}^2 + \widehat{\text{osc}}_k^2(P_k u;\mathcal{T}_k)\big)$$

with $\tilde{\xi}_0^2 \in (0, \frac{1}{2})$. *Denote* $\tilde{\theta} = \theta_*(1-2\tilde{\xi}_0^2)^{\frac{1}{2}}$, *then the set* $\mathcal{T}_k \backslash (\mathcal{T}_{k,*} \cap \mathcal{T}_k)$ *satisfies the following inequality*

$$\widehat{\eta}_k(P_k u; \mathcal{T}_k \backslash (\mathcal{T}_{k,*} \cap \mathcal{T}_k)) \geq \tilde{\theta}\widehat{\eta}_k(P_k u; \mathcal{T}_k).$$

3. Adaptive multigrid method for eigenvalue problem

In this section, we will design a type of adaptive multigrid method for the following eigenvalue problem based on the multilevel correction scheme and adaptive multigrid method.

$$\left\{\begin{array}{rcll} Lu := -\nabla \cdot (\mathcal{A}\nabla u) + \phi u &=& \lambda u & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{10}$$

The corresponding weak form can be written as: Find $(\lambda, u) \in \mathbb{R} \times V$ such that

$$a(u, v) = \lambda(u, v), \quad \forall v \in V. \tag{11}$$

As we know, the eigenvalue problem (11) has an eigenvalue sequence (see [3, 11]):

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \ \lim_{i\to\infty} \lambda_i = \infty$$

and the corresponding eigenfunctions

$$u_1, u_2, \cdots, u_i, \cdots,$$

where $(u_i, u_j) = \delta_{ij}$.

The following property of eigenvalue and eigenfunction approximation is useful (see [3, 44]).

Lemma 3.1. *Let* (λ, u) *be an eigenpair of (11). For any* $w \in V \setminus \{0\}$, *we have*

$$\frac{a(w,w)}{b(w,w)} - \lambda = \frac{a(w-u, w-u)}{b(w,w)} - \lambda\frac{b(w-u,w-u)}{b(w,w)}. \tag{12}$$

The standard finite element method for (11) is to solve the following eigenvalue problem: Find $(\bar{\lambda}_k, \bar{u}_k) \in \mathbb{R} \times V_k$ such that

$$a(\bar{u}_k, v_k) = \bar{\lambda}_k(\bar{u}_k, v_k), \quad \forall v_k \in V_k. \tag{13}$$

From [3, 11], the discrete eigenvalue problem (13) has an eigenvalue sequence

$$0 < \bar{\lambda}_{k,1} \leq \bar{\lambda}_{k,2} \leq \cdots \leq \bar{\lambda}_{k,N_k}$$

and the corresponding eigenfunctions

$$\bar{u}_{k,1}, \bar{u}_{k,2}, \cdots, \bar{u}_{k,N_k},$$

where $(\bar{u}_{k,i}, \bar{u}_{k,j}) = \delta_{i,j}, 1 \leq i, j \leq N_k$ (N_k is the dimension of the finite element space V_k).

Let $M(\lambda_i)$ denote the unit ball in the eigenfunction set corresponding to the eigenvalue λ_i which is defined by

$$M(\lambda_i) = \{w \in V : w \text{ is an eigenfunction of (11) corresponding to } \lambda_i, \|w\|_0 = 1\}. \tag{14}$$

Denote

$$\delta_k(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v_k \in V_k} \|w - v_k\|_{a,\Omega}$$

and

$$\eta_a(V_k) = \sup_{f \in L^2(\Omega), \|f\|_{0,\Omega}=1} \inf_{v_k \in V_k} \|L^{-1}f - v_k\|_{a,\Omega}.$$

For the eigenpair approximation by the finite element method, there exists the following lemma (see [3, 11]).

Lemma 3.2. *There exists the exact eigenpair (λ_i, u_i) of (11) such that each approximate eigenpair $(\bar{\lambda}_{k,i}, \bar{u}_{k,i})$ has the following estimates*

$$\|u_i - \bar{u}_{k,i}\|_{a,\Omega} \lesssim \delta_k(\lambda_i), \tag{15}$$

$$\|u_i - \bar{u}_{k,i}\|_{0,\Omega} \lesssim \eta_a(V_k)\|u_i - \bar{u}_{k,i}\|_{a,\Omega}, \tag{16}$$

$$|\lambda_i - \bar{\lambda}_{k,i}| \lesssim \|u_i - \bar{u}_{k,i}\|^2_{a,\Omega}. \tag{17}$$

3.1. Adaptive multigrid method for eigenvalue problem. In this subsection, we design a new type of adaptive multigrid method for eigenvalue problem (11) based on the combination of the multilevel correction method and adaptive refinement technique.

According to the element residual $\widehat{\mathcal{R}}_T(u_k)$ and the jump residual $\widehat{J}_E(u_k)$ of the boundary value problem (1), we define the element residual and the jump residual of eigenvalue problem (13) as follows:

$$\begin{aligned} \mathcal{R}_T(\lambda_k, u_k) &:= \lambda_k u_k - \phi u_k + \nabla\cdot(\mathcal{A}\nabla u_k) \quad \text{in } T \in \mathcal{T}_k, \\ \mathcal{J}_E(u_k) &:= -\mathcal{A}\nabla u_k^+ \cdot \nu^+ - \mathcal{A}\nabla u_k^- \cdot \nu^- := [[\mathcal{A}\nabla u_k]]_E \cdot \nu_E \quad \text{on } E \in \mathcal{E}_k. \end{aligned}$$

For each element $T \in \mathcal{T}_k$, the local error indicator and oscillation of eigenvalue problem (13) are defined by

$$\eta_k^2(\lambda_k, u_k; T) := h_T^2\|\mathcal{R}_T(\lambda_k, u_k)\|^2_{0,T} + \sum_{E\in\mathcal{E}_k, E\subset\partial T} h_E\|\mathcal{J}_E(u_k)\|^2_{0,E}, \tag{18}$$

$$\begin{aligned} \mathrm{osc}_k^2(\lambda_k, u_k; T) :=& h_T^2\|(I - P_T)\mathcal{R}_T(\lambda_k, u_k)\|^2_{0,T} \\ &+ \sum_{E\in\mathcal{E}_k, E\subset\partial T} h_E\|(I - P_E)\mathcal{J}_E(u_k)\|^2_{0,E}. \end{aligned} \tag{19}$$

Then on a subset $\omega \subset \mathcal{T}_k$, we define the error indicator $\eta_k(\lambda_k, u_k; \omega)$ and oscillation $\mathrm{osc}_k(\lambda_k, u_k; \omega)$ by

$$\eta_k^2(\lambda_k, u_k; \omega) := \sum_{T\in\omega} \eta_k^2(\lambda_k, u_k; T), \quad \mathrm{osc}_k^2(\lambda_k, u_k; \omega) := \sum_{T\in\omega} \mathrm{osc}_k^2(\lambda_k, u_k; T). \tag{20}$$

Similarly, we also introduce some modules of our adaptive multigrid algorithm for eigenvalue problem:

- $(\mu, w) = \mathrm{EG\_SOLVE}(V_k)$: Solve eigenvalue problem (11) in the finite element space V_k and output the discrete approximation (μ, w).
- $\{\eta_k(\lambda_k, u_k; T)\}_{T\in\mathcal{T}_k} = \mathrm{EG\_ESTIMATE}(\lambda_k, u_k, \mathcal{T}_k)$: Compute the error indicator on each element.
- $\mathcal{M}_k = \mathrm{EG\_MARK}(\theta, \eta_k(\lambda_k, u_k; T), \mathcal{T}_k)$: Construct a minimal subset $\mathcal{M}_k$ from $\mathcal{T}_k$ by filtrating relevant elements in $\mathcal{T}_k$ such that

$$\eta_k(\lambda_k, u_k; \mathcal{M}_k) \geq \theta\eta_k(\lambda_k, u_k; \mathcal{T}_k). \tag{21}$$

Then the adaptive multigrid method for eigenvalue problem (13) is defined as follows.

Adaptive Algorithm 2

Given a parameter $\theta \in (0, 1)$. Generate a coarse triangulation $\mathcal{T}_H$ on the computing domain Ω and construct the corresponding finite element space V_H. Pick up an initial mesh $\mathcal{T}_1$ which is produced by refining $\mathcal{T}_H$ by the regular way. Then build the initial finite element space V_1 on the triangulation $\mathcal{T}_1$. Let $k := 1$ and do the following loops:

(1) $(\lambda_k, u_k) = \begin{cases} \text{EG\_SOLVE}(V_1), & \text{when } k = 1; \\ \text{EG\_SOLVE}(V_H \oplus \text{span}\{\breve{u}_k\}), & \text{when } k > 1; \end{cases}$
(2) $\{\eta_k(\lambda_k, u_k; T)\}_{T\in\mathcal{T}_k} = \text{EG\_ESTIMATE}(\lambda_k, u_k, \mathcal{T}_k)$;
(3) $\mathcal{M}_k = \text{EG\_MARK}(\theta, \eta_k(\lambda_k, u_k; T), \mathcal{T}_k)$;
(4) $(\mathcal{T}_{k+1}, V_{k+1}) = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$;
(5) (a) set $u_{k+1}^{(0)} = u_k$;
 (b) For $\ell = 0, \cdots, p-1$:
 $u_{k+1}^{(\ell+1)} = \text{MGBVP\_SOLVE}(\lambda_k u_k, u_{k+1}^{(\ell)}, V_{k+1})$;
 (c) Set $\breve{u}_{k+1} = u_{k+1}^{(p)}$;
(6) Set $k := k + 1$ and go to step 1 until stop.

The enriched space $V_H \oplus \text{span}\{\breve{u}_k\}$ plays an important role in our algorithm, which can not only keep the accuracy of the H_1-norm estimate of the approximate eigenfunction, but also give a better L^2-norm estimate (see e.g. [24, 42]).

In the following analysis, we just need the following crude a priori error estimates for approximate eigenpair (λ_k, u_k) which are stated as follows.

Lemma 3.3. *For the obtained approximate eigenpair* (λ_k, u_k) $(k = 1, 2, \cdots)$ *after each adaptive step in Adaptive Algorithm 2, the following estimates hold*

$$\|u - u_k\|_{a,\Omega} \lesssim \delta_H(\lambda), \tag{22}$$

$$\|u - u_k\|_{0,\Omega} \lesssim \eta_a(V_H)\|u - u_k\|_{a,\Omega}, \tag{23}$$

$$|\lambda - \lambda_k| \lesssim \|u - u_k\|_{a,\Omega}^2. \tag{24}$$

Proof. Based on the finite element error estimate presented in Lemma 3.2, the approximate eigenpair (λ_k, u_k) has the following estimates

$$|\lambda - \lambda_k| \lesssim \|u - u_k\|_{a,\Omega}^2, \tag{25}$$

$$\begin{aligned} \|u - u_k\|_{a,\Omega} &\lesssim \inf_{v_k\in V_H\oplus\text{span}\{\breve{u}_k\}} \|u - v_k\|_{a,\Omega} \\ &\lesssim \inf_{v_k\in V_H} \|u - v_k\|_{a,\Omega} \lesssim \delta_H(\lambda) \end{aligned} \tag{26}$$

and

$$\|u - u_k\|_{0,\Omega} \lesssim \eta_a(V_H \oplus \text{span}\{\breve{u}_k\})\|u - u_k\|_{a,\Omega} \le \eta_a(V_H)\|u - u_k\|_{a,\Omega}. \tag{27}$$

Then we complete the proof. □

3.2. Reliability and efficiency of the a posteriori error estimator for eigenvalue problem. In this subsection, we will show the reliability and efficiency of the a posteriori error estimator for eigenvalue problem defined in (20). In order to derive the theoretical results and also to analyze the convergence and optimal complexity of Adaptive Algorithm 2, we establish the relationship between the solutions of eigenvalue problem (11) and source problem (4) firstly.

Let $w^k \in V$ be the exact solution of the following equation: Find $w^k \in V$ such that

$$a(w^k, v) = (\lambda_k u_k, v) \quad \forall v \in V. \tag{28}$$

Denote

$$\widetilde{u}_k = P_k w^{k-1}, \tag{29}$$

then we obtain the following theorem.

Theorem 3.1. *Assume the adaptive multigrid iteration for boundary value problem*

$$u_k^{(\ell+1)} = \text{MGBVP\_SOLVE}(\lambda_{k-1}u_{k-1}, u_k^{(\ell)}, V_k) \tag{30}$$

has the following error reduction rate:

$$\|\widetilde{u}_k - u_k^{(\ell+1)}\|_{a,\Omega} \leq \nu\|\widetilde{u}_k - u_k^{(\ell)}\|_{a,\Omega}. \tag{31}$$

Then the following estimate holds

$$\begin{aligned} \|u-u_k\|_{a,\Omega} &= \|w^k - P_k w^k\|_{a,\Omega} \\ &\quad +\mathcal{O}(r(V_H,\nu))(\|u-u_{k-1}\|_{a,\Omega} + \|u-u_k\|_{a,\Omega}), \end{aligned} \tag{32}$$

$$\begin{aligned} \|u-u_k\|_{a,\Omega} &= \|w^{k-1} - P_k w^{k-1}\|_{a,\Omega} \\ &\quad +\mathcal{O}(r(V_H,\nu))(\|u-u_{k-1}\|_{a,\Omega} + \|u-u_k\|_{a,\Omega}) \end{aligned} \tag{33}$$

with $r(V_H,\nu) = \eta_a(V_H) + \nu^p$.

Proof. $u - u_k$ can be decomposed as follows

$$u - u_k = u - w^k + w^k - P_k w^k + P_k w^k - P_k w^{k-1} + P_k w^{k-1} - u_k.$$

For the first part, associating with (11) and (28), we have

$$\begin{aligned} \|u-w^k\|_a^2 &= a(u-w^k, u-w^k) \\ &= \big(\lambda u - \lambda_k u_k, u - w^k\big) \\ &\lesssim (|\lambda-\lambda_k| + \|u-u_k\|_{0,\Omega})\|u-w^k\|_{a,\Omega} \\ &\lesssim \eta_a(V_H)\|u-u_k\|_{a,\Omega}\|u-w^k\|_{a,\Omega}. \end{aligned} \tag{34}$$

Hence there holds

$$\|u-w^k\|_{a,\Omega} \lesssim \eta_a(V_H)\|u-u_k\|_{a,\Omega}. \tag{35}$$

With regard to the third part, referring to (6) and the proved result (35), we have the following estimates

$$\begin{aligned} \|P_k(w^k - w^{k-1})\|_{a,\Omega} &\leq \|u-w^k\|_{a,\Omega} + \|u-w^{k-1}\|_{a,\Omega} \\ &\lesssim \eta_a(V_H)\big(\|u-u_k\|_{a,\Omega} + \|u-u_{k-1}\|_{a,\Omega}\big). \end{aligned} \tag{36}$$

For the last term, since $\breve{u}_k - u_k \in V_H \oplus \text{span}\{\breve{u}_k\}$, we have

$$\begin{aligned} &\|P_k w^{k-1} - u_k\|_a^2 = a(P_k w^{k-1} - u_k, P_k w^{k-1} - u_k) \\ =&a(P_k w^{k-1} - u_k, P_k w^{k-1} - \breve{u}_k) + a(P_k w^{k-1} - u_k, \breve{u}_k - u_k) \\ =&a(P_k w^{k-1} - u_k, P_k w^{k-1} - \breve{u}_k) + (\lambda_{k-1}u_{k-1} - \lambda_k u_k, \breve{u}_k - u_k) \\ \lesssim&\|P_k w^{k-1} - u_k\|_a\|P_k w^{k-1} - \breve{u}_k\|_a \\ &+ \eta_a(V_H)(\|u-u_k\|_{a,\Omega} + \|u-u_{k-1}\|_{a,\Omega})\|\breve{u}_k - u_k\|_{a,\Omega} \\ \lesssim&\|P_k w^{k-1} - u_k\|_a\|P_k w^{k-1} - \breve{u}_k\|_a + \eta_a(V_H)(\|u-u_k\|_{a,\Omega} + \|u-u_{k-1}\|_{a,\Omega}) \\ &(\|P_k w^{k-1} - \breve{u}_k\|_{a,\Omega} + \|u_k - P_k w^{k-1}\|_a). \end{aligned} \tag{37}$$

From the error reduction rate presented in (31), we can derive

$$\begin{aligned} \|P_k w^{k-1} - \breve{u}_k\|_a &\leq \nu^p\|P_k w^{k-1} - u_{k-1}\|_a \\ &\leq \nu^p(\|P_k w^{k-1} - P_k u\|_a + \|P_k u - u\|_a + \|u-u_{k-1}\|_a) \\ &\leq \nu^p(\|u-w^{k-1}\|_a + \|u-u_k\|_a + \|u-u_{k-1}\|_a) \\ &\leq \nu^p(1 + C\eta_a(V_H))(\|u-u_k\|_a + \|u-u_{k-1}\|_a). \end{aligned} \tag{38}$$

Combing (37) and (38) leads to

$$\begin{aligned}\|P_kw^{k-1}-u_k\|_a^2 &\lesssim (\nu^p+\eta_a(V_H))\big(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}\big)\|P_kw^{k-1}-u_k\|_a\\ &\quad+\nu^p\eta_a(V_H)\big(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}\big)^2 \qquad (39)\end{aligned}$$

Thus the following estimate holds

$$\|P_kw^{k-1}-u_k\|_{a,\Omega}\lesssim(\nu^p+\eta_a(V_H))\big(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}\big). \qquad (40)$$

Using (35), (36) and (40), we can derive (32).

The second identity (33) can be proved by the same token using decomposition of $u-u_k$

$$u-u_k=u-w^{k-1}+w^{k-1}-P_kw^{k-1}+P_kw^{k-1}-u_k.$$

So we complete the proof. □

Theorem 3.1 establishes a relationship between the error estimate of eigenvalue problem and boundary value problem. Since the difference is a higher order term and the theoretical results of boundary value problem has already been well analyzed, we can derive the conclusions of adaptive multigrid method for eigenvalue problem by following the procedure of linear elliptic boundary value problem [10], and this technique can also be found in [15, 22].

Similarly, the following theorem can also be proved by combining the definitions of error indicators, Sobolev trace theorem and the inverse inequality of the finite element method.

Theorem 3.2. *We have following properties for the error estimator*

$$\eta_k(\lambda_k,u_k;\mathcal{T}_k)=\widehat{\eta}_k(P_kw^{k-1};\mathcal{T}_k)+\mathcal{O}\big(r(V_H,\nu)\big)(\|u-u_{k-1}\|_{a,\Omega}+\|u-u_k\|_{a,\Omega}), \qquad (41)$$

$$\eta_k(\lambda_k,u_k;\mathcal{T}_k)=\widehat{\eta}_k(P_kw^{k};\mathcal{T}_k)+\mathcal{O}\big(r(V_H,\nu)\big)(\|u-u_{k-1}\|_{a,\Omega}+\|u-u_k\|_{a,\Omega}). \qquad (42)$$

Proof. From the definitions of $\eta_k(\lambda_k,u_k;T)$ for eigenvalue problem and $\widehat{\eta}_k(P_kw^k;T)$ for boundary value problem, we have

$$\begin{aligned}&|\eta_k(\lambda_k,u_k;T)-\widehat{\eta}_k(P_kw^{k-1};T)|\\ =&\bigg|\Big(h_T^2\|\lambda_ku_k-\phi u_k+\nabla\cdot(\mathcal{A}\nabla u_k)\|_{0,T}^2+\sum_{e\in\mathcal{E}_k,e\subset\partial T}h_e\|[\mathcal{A}\nabla u_k]_e\cdot\nu_e\|_{0,e}^2\Big)^{\frac12}\\ &-\Big(h_T^2\|\lambda_{k-1}u_{k-1}-\phi\widetilde{u}_k+\nabla\cdot(\mathcal{A}\nabla\widetilde{u}_k)\|_{0,T}^2+\sum_{e\in\mathcal{E}_k,e\subset\partial T}h_e\|[\mathcal{A}\nabla\widetilde{u}_k]_e\cdot\nu_e\|_{0,e}^2\Big)^{\frac12}\bigg|\\ \le&\bigg\{\Big(h_T^2\|\lambda_ku_k-\lambda_{k-1}u_{k-1}-\phi(u_k-\widetilde{u}_k)+\nabla\cdot(\mathcal{A}\nabla(u_k-\widetilde{u}_k))\|_{0,T}\Big)^2\\ &+h_e\sum_{e\in\mathcal{E}_k,e\subset\partial T}\Big(\|[\mathcal{A}\nabla u_k]_e\cdot\nu_e-[\mathcal{A}\nabla\widetilde{u}_k]_e\cdot\nu_e\|_{0,e}\Big)^2\bigg\}^{\frac12}. \qquad (43)\end{aligned}$$

It is obvious that the inverse estimate implies

$$\|\nabla\cdot(\mathcal{A}\nabla v_k)\|_{0,T}\lesssim h_T^{-1}\|\nabla v_k\|_{0,T},\quad \forall T\in\mathcal{T}_h, v_k\in V_k. \qquad (44)$$

From the inverse estimate and the trace inequality

$$\|v\|_{0,\partial T}\lesssim h_T^{-1/2}\|v\|_{0,T}+h_T^{1/2}\|v\|_{1,T},\quad \forall v\in H^1(T),\ T\in\mathcal{T}_k,$$

we have

$$h_e\|[\mathcal{A}\nabla v_k]_e\cdot\nu_e\|_{0,e}^2\lesssim\|\nabla v_k\|_{0,T}^2\lesssim\|v_k\|_{a,T}^2,\quad \forall v_k\in V_{h_k}. \qquad (45)$$

Combining (43)-(45) with (40) leads to

$$\begin{aligned}&|\eta_k(\lambda_k,u_k;T)-\widetilde{\eta}_k(P_kw_{k-1};T)|\\ \lesssim\ & h_T\|\lambda_ku_k-\lambda_{k-1}u_{k-1}\|_{0,T}+\|u_k-\widetilde{u}_k\|_{a,T}.\end{aligned}\tag{46}$$

From Lemma 3.2 and (46), there holds

$$\begin{aligned}&|\eta_k(\lambda_k,u_k;\mathcal{T}_k)-\widetilde{\eta}_k(P_kw_{k-1};\mathcal{T}_k)|\\ =&\big|\big(\sum_{T\in\mathcal{T}_k}\eta_k^2(\lambda_k,u_k;T)\big)^{1/2}-\big(\sum_{T\in\mathcal{T}_k}\widetilde{\eta}_k^2(P_kw_{k-1};T)\big)^{1/2}\big|\\ \lesssim&\big(\sum_{T\in\mathcal{T}_k}\big(\eta_k(\lambda_k,u_k;T)-\widetilde{\eta}_k(P_kw_{k-1};T)\big)^2\big)^{1/2}\\ \lesssim& r(V_H,\nu)(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}).\end{aligned}$$

This is the desired result (41). The result (42) can be derived similarly and we complete the proof. □

Theorem 3.3. *We have following properties for the oscillation*

$$\begin{aligned}\mathrm{osc}_k(\lambda_k,u_k;\mathcal{T}_k)=&\widehat{\mathrm{osc}}_k(P_kw^{k-1};\mathcal{T}_k)\\ &+\mathcal{O}\big(r(V_H,\nu)\big)(\|u-u_{k-1}\|_{a,\Omega}+\|u-u_k\|_{a,\Omega}),\end{aligned}\tag{47}$$

$$\begin{aligned}\mathrm{osc}_k(\lambda_k,u_k;\mathcal{T}_k)=&\widehat{\mathrm{osc}}_k(P_kw^{k};\mathcal{T}_k)\\ &+\mathcal{O}\big(r(V_H,\nu)\big)(\|u-u_{k-1}\|_{a,\Omega}+\|u-u_k\|_{a,\Omega}).\end{aligned}\tag{48}$$

Proof. Following the similar procedure of Theorem 3.2 and the definition of oscillation, it is easy to derive the desired estimates. □

Based on Theorems 3.1-3.3, we can obtain the following reliability and efficiency of the a posteriori error estimator for eigenvalue problem by applying Lemma 2.1.

Theorem 3.4. *When $r(V_H,\nu)$ is small enough, there exist constants C_{u} and C_ℓ independent of mesh index k such that*

$$\|u-u_k\|_{a,\Omega}^2\le C_{\mathrm{u}}\eta_k^2(\lambda_k,u_k;\mathcal{T}_k)+\mathcal{O}(r^2(V_H,\nu))\|u-u_{k-1}\|_{a,\Omega}^2\tag{49}$$

and

$$\begin{aligned}C_\ell\eta_k^2(\lambda_k,u_k;\mathcal{T}_k)\le&\|u-u_k\|_{a,\Omega}^2+\mathrm{osc}_k^2(\lambda_k,u_k;\mathcal{T}_k)\\ &+\mathcal{O}(r^2(V_H,\nu))\|u-u_{k-1}\|_{a,\Omega}^2.\end{aligned}\tag{50}$$

Proof. Since w^{k-1} is the exact solution of boundary value problem, from Lemma 2.1, there hold

$$\|w^{k-1}-P_kw^{k-1}\|_{a,\Omega}\le\widehat{C}_u\widehat{\eta}_k(P_kw^{k-1},\mathcal{T}_k)$$

and

$$\widehat{C}_\ell\widehat{\eta}_k^2(P_kw^{k-1},\mathcal{T}_k)\le\|w^{k-1}-P_kw^{k-1}\|_{a,\Omega}^2+\widehat{osc}_k^2(P_kw^{k-1},\mathcal{T}_k).$$

Then from Theorems 3.1-3.3, we can get the desired results. □

4. Convergence and complexity analysis of adaptive multigrid algorithm

In this section, we will analyze the convergence and complexity property of Adaptive Algorithm 2. In the rest of this paper, we assume the mesh size H and ν^p are small enough such that

$$r(V_H,\nu)\|u-u_{k-1}\|_{a,\Omega}^2 \lesssim \|u-u_k\|_{a,\Omega}^2, \quad \text{for } k\geq 2. \tag{51}$$

This assumption means the initial mesh size h_1 should be small enough that the error will not change too much after each adaptive step. In addition, in order to meet (51), we may need to execute more than one time multigrid iteration step ($p\geq 1$). In our numerical experiment, two or three times iteration steps are enough to derive the optimal accuracy due to the high efficiency of multigrid method.

4.1. Convergence of adaptive multigrid algorithm.

In this subsection, we will show the convergence of Adaptive Algorithm 2 for eigenvalue problem based on the existing results for elliptic boundary value equation and Theorems 3.1, 3.2.

Theorem 4.1. *For the finite element approximate eigenfunction sequence $\{u_k\}$ produced by Adaptive Algorithm 2, there exist constants $\gamma>0$ and $\alpha\in(0,1)$, depending on the shape regularity of mesh and the refinement parameter θ, such that when $r(V_H,\nu)$ is small enough, there holds*

$$\begin{aligned}&\|u-u_k\|_{a,\Omega}^2+\gamma\eta_k^2(\lambda_k,u_k;\mathcal{T}_k)\\ \leq&\alpha^2\big(\|u-u_{k-1}\|_{a,\Omega}^2+\gamma\eta_{k-1}^2(\lambda_{k-1},u_{k-1};\mathcal{T}_{k-1})\big).\end{aligned} \tag{52}$$

Proof. From Lemma 2.1, (32) and assumption (51), we can derive the following inequalities

$$\|u-u_{k-1}\|_{a,\Omega}\leq\frac{\|w^{k-1}-P_{k-1}w^{k-1}\|_{a,\Omega}}{1-Cr^{\frac12}(V_H,\nu)}\leq\frac{\hat{C}_{\mathrm{u}}^{\frac12}\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{T}_{k-1})}{1-Cr^{\frac12}(V_H,\nu)}. \tag{53}$$

By marking strategy (21) and the proof of Theorem 3.2, we can derive

$$\begin{aligned}&\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{M}_{k-1})\\ \geq\ &\eta_{k-1}(\lambda_{k-1},u_{k-1};\mathcal{M}_{k-1})-C_\eta r^{\frac12}(V_H,\nu)\|u-u_{k-1}\|_{a,\Omega}\\ \geq\ &\theta\eta_{k-1}(\lambda_{k-1},u_{k-1};\mathcal{T}_{k-1})-C_\eta r^{\frac12}(V_H,\nu)\|u-u_{k-1}\|_{a,\Omega}\\ \geq\ &\theta\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{T}_{k-1})-(1+\theta)C_\eta r^{\frac12}(V_H,\nu)\|u-u_{k-1}\|_{a,\Omega}.\end{aligned} \tag{54}$$

Combing (53) and (54) leads to

$$\begin{aligned}&\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{M}_{k-1})\\ \geq&\Big[\theta-\frac{\hat{C}_{\mathrm{u}}^{\frac12}(1+\theta)C_\eta r^{\frac12}(V_H,\nu)}{1-Cr^{\frac12}(V_H,\nu)}\Big]\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{T}_{k-1}).\end{aligned} \tag{55}$$

So when $r(V_H,\nu)$ is small enough, there exists constant $\widehat{\theta}$ satisfying $\widehat{\theta}\in(0,\theta_*)$ and the following inequality holds

$$\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{M}_{k-1})\geq\widehat{\theta}\widehat{\eta}_{k-1}(P_{k-1}w^{k-1};\mathcal{T}_{k-1}). \tag{56}$$

From Theorem 2.1, there exist constants $\hat{\gamma}>0$ and $\hat{\xi}\in(0,1)$ such that

$$\begin{aligned}&\|w^{k-1}-P_kw^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_k^2(P_kw^{k-1};\mathcal{T}_k)\\ \leq\ &\hat{\xi}^2\big(\|w^{k-1}-P_{k-1}w^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_{k-1}^2(P_{k-1}w^{k-1};\mathcal{T}_{k-1})\big).\end{aligned} \tag{57}$$

Combing (33), (41), (51) and Young inequality leads to

$$
\begin{aligned}
& \|u-u_k\|_{a,\Omega}^2+\hat{\gamma}\eta_k^2(\lambda_k,u_k;\mathcal{T}_k)\\
\leq\;& (1+\delta_1)\big(\|w^{k-1}-P_kw^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_k^2(P_kw^{k-1};\mathcal{T}_k)\big)\\
& +C\delta_1^{-1}r^2(V_H,\nu)\big(\|u-u_k\|_{a,\Omega}^2+\|u-u_{k-1}\|_{a,\Omega}^2\big)\\
\leq\;& (1+\delta_1)\big(\|w^{k-1}-P_kw^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_k^2(P_kw^{k-1};\mathcal{T}_k)\big)\\
& +C\delta_1^{-1}r(V_H,\nu)\big(\|u-u_k\|_{a,\Omega}^2+\hat{\gamma}\eta_k^2(u_k;\mathcal{T}_k)\big),
\end{aligned}
$$

which implies the following inequality

$$
\begin{aligned}
& \|u-u_k\|_{a,\Omega}^2+\hat{\gamma}\eta_k^2(\lambda_k,u_k;\mathcal{T}_k)\\
\leq\;& \frac{1+\delta_1}{1-C\delta_1^{-1}r(V_H,\nu)}\big(\|w^{k-1}-P_kw^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_k^2(P_kw^{k-1};\mathcal{T}_k)\big)\\
\leq\;& \frac{(1+\delta_1)\hat{\xi}^2}{1-C\delta_1^{-1}r(V_H,\nu)}\big(\|w^{k-1}-P_{k-1}w^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_k^2(P_{k-1}w^{k-1};\mathcal{T}_{k-1})\big). \qquad (58)
\end{aligned}
$$

By using the similar argument, we can derive

$$
\begin{aligned}
& \|w^{k-1}-P_{k-1}w^{k-1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_{k-1}^2(P_{k-1}w^{k-1};\mathcal{T}_{k-1})\\
\leq\;& \big(1+\delta_1+C\delta_1^{-1}r(V_H,\nu)\big)\big(\|u-u_{k-1}\|_{a,\Omega}^2+\hat{\gamma}\eta_{k-1}^2(\lambda_{k-1},u_{k-1};\mathcal{T}_{k-1})\big). \qquad (59)
\end{aligned}
$$

Combining (58) and (59) leads to the following inequality

$$
\|u-u_k\|_{a,\Omega}^2+\gamma\eta_k^2(\lambda_k,u_k;\mathcal{T}_k)\leq\alpha^2\big(\|u-u_{k-1}\|_{a,\Omega}^2+\gamma\eta_{k-1}^2(\lambda_{k-1},u_{k-1};\mathcal{T}_{k-1})\big)
$$

with

$$
\alpha^2:=\frac{(1+\delta_1)\big(1+\delta_1+C\delta_1^{-1}r(V_H,\nu)\big)\hat{\xi}^2}{1-C\delta_1^{-1}r(V_H,\nu)},\quad \gamma:=\hat{\gamma}. \qquad (60)
$$

Then the desired result (52) can be deduced by choosing δ_1 small enough such that $\alpha<1$ and the proof is completed. □

4.2. Complexity analysis. In this subsection, we will prove the optimal complexity of Adaptive Algorithm 2. As in the normal analysis of AFEM for elliptic boundary value problems, in order to state the result of the complexity estimate, we introduce a function approximation class as follows (cf. [10])

$$
\mathcal{A}^s:=\big\{v\in H_0^1(\Omega):|v|_s<\infty\big\},
$$

where

$$
|v|_s=\sup_{\varepsilon>0}\varepsilon\inf_{\{\mathcal{T}_1\leq\mathcal{T}_\varepsilon:\inf\limits_{(\lambda_\varepsilon,u_\varepsilon)}(\|v-u_\varepsilon\|_{a,\Omega}^2+\mathrm{osc}_\varepsilon^2(\lambda_\varepsilon,u_\varepsilon;\mathcal{T}_\varepsilon))^{1/2}\leq\varepsilon\}}(\#\mathcal{T}_\varepsilon-\#\mathcal{T}_1)^s
$$

and $\mathcal{T}_1\leq\mathcal{T}_\varepsilon$ means $\mathcal{T}_\varepsilon$ is a conforming refinement of $\mathcal{T}_1$. We use $\#\mathcal{T}$ to denote the number of elements in the mesh $\mathcal{T}$. Hence the symbol $\mathcal{A}^s$ is the class of functions that can be approximated with a given tolerance ε by continuous piecewise polynomial functions over a partition $\mathcal{T}_\varepsilon$ with the number of degrees of freedom $\#\mathcal{T}_\varepsilon-\#\mathcal{T}_1\lesssim\varepsilon^{-1/s}|v|_s^{1/s}$.

Notice that the convergence result presented in Theorem 4.1 is the same as that in [10, 15]. By using the same technique, we can prove that Adaptive Algorithm 2 has the following optimal complexity. Please refer to papers [10, 15] for the detailed proof.

Theorem 4.2. *Let $u \in (H_0^1(\Omega)\cap \mathcal{A}^s)$ be the eigenfunction of (11) and $\{(\lambda_k, u_k)\}$ be the finite element approximations corresponding to the sequence of spaces $\{V_k\}$ produced by Adaptive Algorithm 2. Then under the assumption (51), the ℓ-th iterate solution of Adaptive Algorithm 2 satisfies the optimal bounds*

$$\|u-u_\ell\|_{a,\Omega}^2 + \mathrm{osc}_\ell^2(\lambda_\ell, u_\ell; \mathcal{T}_\ell) \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_1)^{-2s},$$

where the hidden constant depends on the discrepancy between θ and θ_*.

Now we come to briefly estimate the computational work of Adaptive Algorithm 2. Here we have to use additionally, that the sequence of unknowns belongs to a geometric progression:

$$N_k < \sigma_0 N_k \le N_{k+1} < \sigma_1 N_k, \quad k = 1, 2, \cdots \tag{61}$$

Theorem 4.3. *Assume eigenvalue problem solving in the coarse spaces V_H and V_1 need work M_H and M_1, respectively, and the work of the multigrid solver in each adaptive space V_k is $O(N_k)$ for $k = 2, 3, \cdots, \ell$. Then the total computational work of Adaptive Algorithm 2 can be bounded by $O\big(M_1 + M_H\log(N_\ell) + N_\ell\big)$ and furthermore $O\big(N_\ell\big)$ provided M_H and M_1 is small enough.*

Proof. Let W denote the whole computational work of Adaptive Algorithm 2, W_k denote the work on the k-th level for $k = 1, \cdots, \ell$. From the definition of Adaptive Algorithm 2 and (61), it follows that

$$\begin{aligned}
W = \sum_{k=1}^{\ell} W_k &= \mathcal{O}\Big(M_1 + \sum_{k=2}^{\ell}(N_k + M_H)\Big)\\
&= \mathcal{O}\Big(M_1 + M_H(\ell-1) + N_\ell \sum_{k=2}^{\ell}\Big(\frac{1}{\sigma_0}\Big)^{(\ell-k)}\Big)\\
&= \mathcal{O}\big(M_1 + M_H\log(N_\ell) + N_\ell\big).
\end{aligned}$$

Thus, the computational work W can be bounded by $O\big(M_1 + M_H\log(N_\ell) + N_\ell\big)$, and moreover, by $\mathcal{O}(N_\ell)$ if M_H and M_1 are small enough. $\square$

5. Numerical experiments

In this section, we present two numerical examples for eigenvalue problem by Adaptive Algorithm 2. In these numerical examples, we set $p = 3$, and each adaptive multigrid iteration step for boundary value problem is executed with one multigrid V-cycle as the basic iteration using two times conjugate gradient smoother. The package ARPACK is called here for the small-scale eigenvalue problems. In this paper, all numerical examples are running on the machine PowerEdge R720 with the Linux system. The machine is equipped with Intel Xeon E5-2620 (2.00GHz) CPU with 72G memory.

Example 1. In the first example, we consider the following eigenvalue problem (see [20]):

$$\left\{\begin{array}{rcll} -\frac{1}{2}\Delta u + \frac{1}{2}|x|^2 u &=& \lambda u & \text{in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{62}$$

where $\Omega = \mathbb{R}^3$ and $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The first eigenvalue and eigenfunction of (62) is $\lambda = 1.5$ and $u = \kappa e^{-|x|^2/2}$, where κ is a nonzero constant. Since the eigenfunction decays exponentially, we choose $\Omega = (-6, 6)^3$.

In this example, we give the numerical results of Adaptive Algorithm 2 for eigenvalue problem (62) with the parameter $\theta = 0.4$. The initial mesh $V_1 = V_H$. In order to show the efficiency of Adaptive Algorithm 2 more clearly, we compare results with those obtained by direct AFEM. Figure 1 shows the initial mesh and the triangulation after 15 adaptive iterations with linear finite element method. Figure 2 gives the corresponding error estimate. It is shown in Figure 2 that the approximate eigenfunction generated by Adaptive Algorithm 2 has the optimal convergence rate which coincides with our theoretical result.

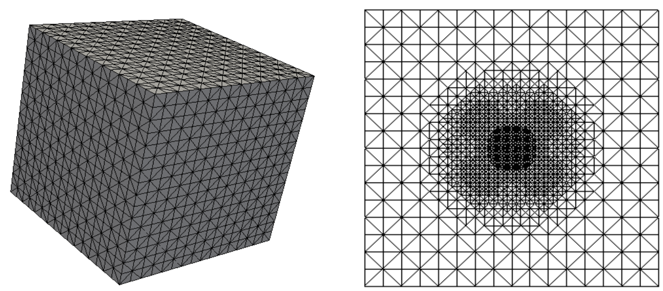

FIGURE 1. The initial mesh and the triangulation after 15 adaptive iterations for Example 1 by Adaptive Algorithm 2 with linear element.

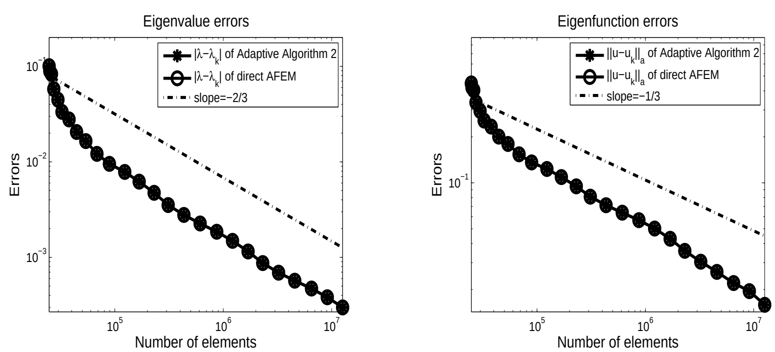

FIGURE 2. Errors of Adaptive Algorithm 2 for Example 1.

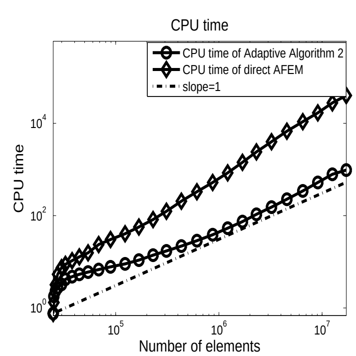

FIGURE 3. CPU time of Adaptive Algorithm 2 for Example 1.

In addition, we presented the CPU time of Adaptive Algorithm 2 and direct AFEM to show the efficiency of Adaptive Algorithm 2. The presented CPU time denotes all the computational time including mesh refinement, assembling matrix and solving process. The corresponding results are presented in Figure 3, which shows that Adaptive Algorithm 2 has a better efficiency than direct AFEM.

Example 2. In the second example, we solve the following Laplace eigenvalue problem:

$$\left\{\begin{array}{rcll} -\nabla\cdot(\mathcal{A}\nabla u)+\phi u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega, \end{array}\right. \tag{63}$$

where $\Omega=(-1,1)^3\setminus[0,1)^3$,

$$\mathcal{A}=\left(\begin{array}{ccc} 1+(x_1-\frac{1}{2})^2 & (x_1-\frac{1}{2})(x_2-\frac{1}{2}) & (x_1-\frac{1}{2})(x_3-\frac{1}{2})\\ (x_2-\frac{1}{2})(x_1-\frac{1}{2}) & 1+(x_2-\frac{1}{2})^2 & (x_2-\frac{1}{2})(x_3-\frac{1}{2})\\ (x_3-\frac{1}{2})(x_1-\frac{1}{2}) & (x_3-\frac{1}{2})(x_2-\frac{1}{2}) & 1+(x_3-\frac{1}{2})^2 \end{array}\right),$$

$\phi=e^{(x_1-\frac{1}{2})(x_2-\frac{1}{2})(x_3-\frac{1}{2})}$. Due to the reentrant corner of Ω, the exact eigenfunction with singularities is expected.

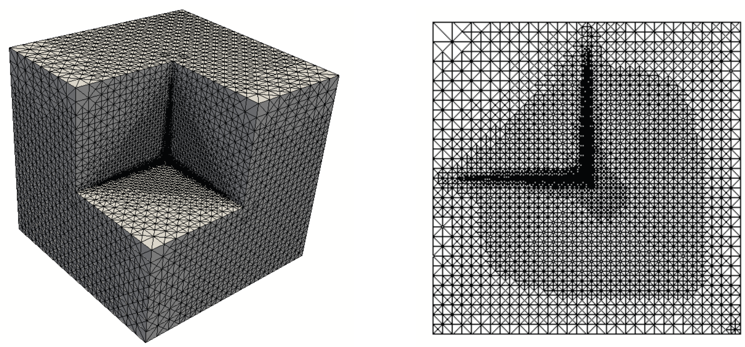

FIGURE 4. The triangulations after 15 adaptive refinements and the corresponding cross section of Adaptive Algorithm 2 for Example 2.

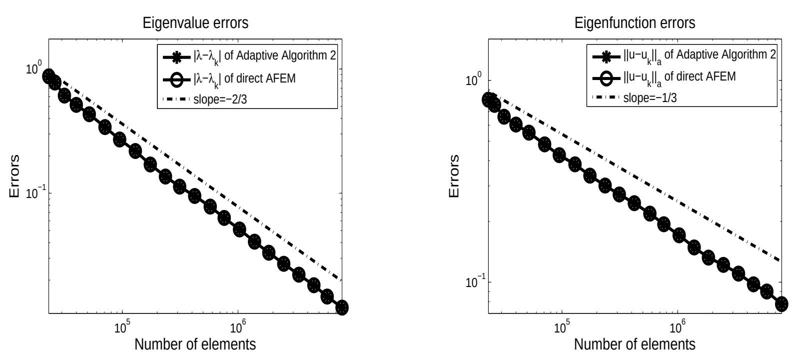

FIGURE 5. Errors of Adaptive Algorithm 2 for Example 2.

Since the exact eigenfunction is not known, an adequately accurate approximate solution on fine finite element space is chosen as the exact one in numerical experiments. In this example, we give numerical results of Adaptive Algorithm 2 with the parameter $\theta=0.4$. The initial mesh $V_1=V_H$. In order to show the efficiency of Adaptive Algorithm 2, we also compare results with those obtained by direct AFEM. Figure 4 shows the triangulation after 15 adaptive iterations with linear

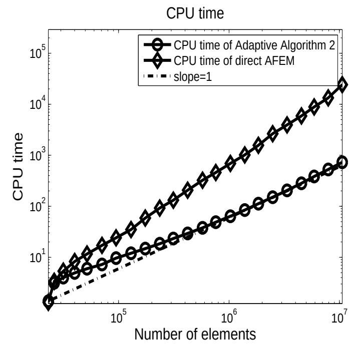

FIGURE 6. CPU time of Adaptive Algorithm 2 for Example 2.

finite element method and the corresponding section along XY plane. Figure 5 gives error estimate. From Figure 5, we can also find that the approximate solution by Adaptive Algorithm 2 has the optimal convergence rate.

In addition, we also presented the CPU time of Adaptive Algorithm 2 and direct AFEM to show the efficiency of Adaptive Algorithm 2. The corresponding results are presented in Figure 6, which shows that Adaptive Algorithm 2 has a better efficiency than direct AFEM.

6. Concluding remarks

In this paper, a type of adaptive multigrid method is proposed for eigenvalue problem based on adaptive multigrid method and recent works on multilevel correction method. The core idea is to transform the eigenvalue problem into a series of elliptic boundary value problems in the sequence of adaptive finite element spaces and some eigenvalue problems in a very low dimensional space. And the involved elliptic boundary value equations are solved by adaptive multigrid method. What's more, the convergence and optimal complexity of the proposed algorithm is verified theoretically and demonstrated numerically. The idea and algorithm in this paper can be further extended to other nonlinear eigenvalue problems such as Kohn-Sham equation.

Acknowledgments

This research was supported by National Science Foundation of China (NSFC 11801021, 11971047).

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