

LOW REGULARITY PRIMAL-DUAL WEAK GALERKIN FINITE ELEMENT METHODS FOR ILL-POSED ELLIPTIC CAUCHY PROBLEMS

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Abstract. A new primal-dual weak Galerkin (PDWG) finite element method is introduced and analyzed for the ill-posed elliptic Cauchy problems with ultra-low regularity assumptions on the exact solution. The Euler-Lagrange formulation resulting from the PDWG scheme yields a system of equations involving both the primal equation and the adjoint (dual) equation. The optimal order error estimate for the primal variable in a low regularity assumption is established. A series of numerical experiments are illustrated to validate effectiveness of the developed theory.

Key words. Primal-dual, finite element method, weak Galerkin, low regularity, elliptic Cauchy equations, ill-posed.

1. Introduction

In this paper we consider the ill-posed elliptic Cauchy model problem: Find an unknown function u satisfying

$$(1) \quad \begin{aligned} -\nabla \cdot (a\nabla u) &= f, & \text{in } \Omega, \\ u &= g_1, & \text{on } \Gamma_D, \\ a\nabla u \cdot \mathbf{n} &= g_2, & \text{on } \Gamma_N, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is an open bounded and connected domain with Lipschitz continuous boundary $\partial\Omega$, Γ_D and Γ_N are two segments of the domain boundary, $f \in L^2(\Omega)$, the Cauchy data $g_1 \in H^{\frac{1}{2}}(\Gamma_D)$ and $g_2 \in (H^{\frac{1}{2}}_0(\Gamma_N))'$, the coefficient tensor $a(x)$ is symmetric, bounded, and uniformly positive definite in the domain Ω , and \mathbf{n} is the unit outward normal vector to Γ_N . The elliptic Cauchy problem is to solve partial differential equations (PDEs) in a domain where over-specified boundary conditions are given on parts of the domain boundary. The elliptic Cauchy problem is also to solve a data completion problem with missing boundary data on the remaining parts of the domain boundary.

The elliptic Cauchy problem arises in science and engineering, e.g., vibration, wave propagation, cardiology, electromagnetic scattering, geophysics, nondestructive testing and steady-state inverse heat conduction. In particular, the Cauchy problem for second order elliptic equations plays an important role in the inverse boundary value problems modeled by elliptic PDEs. Readers are referred to [4, 28, 33, 40, 1, 9, 10, 16, 17, 46, 19, 53, 7, 20, 21] and the references cited therein for details of the elliptic Cauchy problems.

There has been a long history tracing back to Hadamard [25, 43, 27, 26, 24] for the study of the elliptic Cauchy problem (1). When it comes to the case of $\Gamma_D = \Gamma_N$, Hadamard demonstrated the ill-posedness of the problem (1) by constructing an example where the solution does not depend continuously on the Cauchy data. Hadamard and others [2, 29, 30] found that a small perturbation in the data might result in an enormous error in the numerical solution for elliptic Cauchy problem.

The Schwartz reflection principle [23] indicates that in most cases the existence of solutions for the model problem (1) can not be guaranteed for any given Cauchy data g_1 and g_2 . However, [3] showed that the elliptic Cauchy problem (1) has a solution for any given Cauchy data $g_1 \times g_2 \in M$ where M is a dense subset of $H^{\frac{1}{2}}(\Gamma_D) \times [H_{00}^{\frac{1}{2}}(\Gamma_N)]'$. It is well-known that the solution of the elliptic Cauchy problem (1) (if it exists) must be unique, provided that $\Gamma_D \cap \Gamma_N$ is a nontrivial portion of the domain boundary. The Cauchy data is thus assumed to be compatible such that the solution exists. Throughout this paper, we assume that $\Gamma_D \cap \Gamma_N$ is a nontrivial portion of the domain boundary so that the solution (if it exists) of the elliptic Cauchy problem (1) is unique.

In the literature, there are two main numerical strategies developed for the elliptic Cauchy problem: (1) Tikhonov regularization is applied to the problem with missing boundary data to determine the solution; (2) A sequence of well-posed problems in the same equation is iteratively employed to approximate the ill-posed problem. [34] developed the numerical method for the elliptic Cauchy problem based on the tools of boundary integral equations, single-layer potential function and jump relations. [22] introduced an optimization approach based on least squares and Tikhonov regularization techniques. The finite element method based on an optimal control characterization of the Cauchy problem was analyzed in [14]. The stabilized finite element method [12, 11] based on a general framework involving both the original equation and its adjoint equation is applicable to a wide class of ill-posed problems where only weak continuity is necessary. More numerical methods were proposed and analyzed for the elliptic Cauchy problem including the conjugate gradient boundary element method, the boundary knot method, the alternating iterative boundary element method, the moment method, the boundary particle method, the method of level set type, and the method of fundamental solutions [35, 40, 54, 15, 41, 31, 32, 42, 53]. Other theoretical and applied work have also been developed such as regularization methods [45, 18], Steklov-Poincaré theory [6, 44, 5], minimal error methods [38, 39] and quasi-reversibility methods [8].

This paper is devoted to the development of a new primal-dual weak Galerkin finite element method for the elliptic Cauchy model problem (1). The PDWG framework provides mechanisms to enhance the stability of a numerical scheme by combining solutions of the primal and the dual (adjoint) equation. PDWG methods have been successfully applied to solve the second order elliptic equation in non-divergence form [48], the elliptic Cauchy problem [47, 49], the Fokker-Planck type equation [50], the convection diffusion equation [13, 55], and the transport equation [51, 36]. The PDWG method has the following advantages over other existing schemes: (1) it offers a symmetric and well-posed problem for the ill-posed elliptic Cauchy problem; (2) it is consistent in the sense that the system is satisfied by the exact solution (if it exists); (3) it is applicable to a wide class of PDE problems for which no traditional variational formulation is available; and (4) it admits general finite element partitions consisting of arbitrary polygons or polyhedra. The main contribution of this paper lies in two aspects: (1) the development of a new PDWG scheme that admits boundary data with low regularity due to noise or uncertainties; and (2) the establishment of a mathematical convergence theory with optimal order error estimates under low regularity assumptions for the exact solution.

Throughout the paper, we use the standard notations for Sobolev spaces and norms. For any open bounded domain $D \subset \mathbb{R}^d$ with Lipschitz continuous boundary, denote by $\|\cdot\|_{s,D}$, $|\cdot|_{s,D}$ and $(\cdot, \cdot)_{s,D}$ the norm, seminorm and the inner product

in the Sobolev space $H^s(D)$ for $s \geq 0$, respectively. The norms in $H^s(D)$ ($s < 0$) are defined by duality with the norms in $H^{|s|}(D)$. The space $H^0(D)$ coincides with $L^2(D)$, where the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, or when the domain of integration is clear from the context, the subscript D is dropped in the norm and the inner product notation. For convenience, we use “ \lesssim ” to denote “less than or equal to” up to a generic constant which is independent of important parameters such as the mesh size and physical parameters.

The paper is organized as follows. In Section 2, the weak formulation of the elliptic Cauchy model problem is proposed and the discrete weak differential operator is briefly reviewed. In Section 3, the PDWG algorithm for the elliptic Cauchy model problem is introduced. Section 4 demonstrates the solution existence and uniqueness of the proposed PDWG scheme. The error equations for the PDWG scheme are derived in Section 5. In Section 6, the optimal order error estimates are established for the PDWG method in some discrete Sobolev norms. In Section 7, a series of numerical results are reported to demonstrate the effectiveness of the PDWG method developed in the previous sections.

2. Weak Formulations and Discrete Weak Differential Operators

This section will introduce the weak formulation of the elliptic Cauchy model problem (1) and briefly review the discrete weak differential operator [52].

Denote by $\Gamma_N^c = \partial\Omega \setminus \Gamma_N$ and $\Gamma_D^c = \partial\Omega \setminus \Gamma_D$. The weak formulation of the elliptic Cauchy model problem (1) seeks $u \in L^2(\Omega)$ satisfying

$$(2) \quad (u, \nabla \cdot (a\nabla\sigma)) = -(f, \sigma) - \langle g_2, \sigma \rangle_{\Gamma_N} + \langle g_1, a\nabla\sigma \cdot \mathbf{n} \rangle_{\Gamma_D}, \quad \forall \sigma \in W,$$

where $W = \{\sigma \in H^1(\Omega), a\nabla\sigma \in H(\operatorname{div}; \Omega), \sigma|_{\Gamma_N^c} = 0, a\nabla\sigma \cdot \mathbf{n}|_{\Gamma_D^c} = 0\}$.

Let \mathcal{T}_h be a partition of the domain Ω into polygons in 2D or polyhedra in 3D which is shape regular in the sense of [52]. Denote by \mathcal{E}_h the set of all edges or flat faces in \mathcal{T}_h and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges or flat faces. Denote by h_T the meshsize of $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize for the partition \mathcal{T}_h . For the convenience of analysis and without loss of generality, in what follows of this paper, we assume that the coefficient tensor a is piecewise constants with respect to the partition \mathcal{T}_h . The analysis could be easily generalized to the case that the coefficient tensor a is piecewise smooth functions with respect to the partition \mathcal{T}_h .

Let $T \in \mathcal{T}_h$ be a polygonal or polyhedral region with boundary ∂T . For any $\phi \in H^{1-\theta}(T)$ ($0 \leq \theta < 1$) and any polynomial ψ , the following trace inequalities hold true [52]

$$(3) \quad \|\phi\|_{\partial T}^2 \lesssim h_T^{-1} \|\phi\|_T^2 + h_T^{1-2\theta} \|\phi\|_{1-\theta, T}^2, \quad \|\psi\|_{\partial T}^2 \lesssim h_T^{-1} \|\psi\|_T^2.$$

A weak function on $T \in \mathcal{T}_h$ is denoted by a triplet $\sigma = \{\sigma_0, \sigma_b, \sigma_n\}$ such that $\sigma_0 \in L^2(T)$, $\sigma_b \in L^2(\partial T)$ and $\sigma_n \in L^2(\partial T)$. The first and the second components, namely σ_0 and σ_b , represent the values of σ in the interior and on the boundary of T respectively. The third component σ_n can be understood as the value of $a\nabla\sigma \cdot \mathbf{n}$ on ∂T , where \mathbf{n} is the unit outward normal vector on ∂T . Note that σ_b and σ_n may not necessarily be the traces of σ_0 and $a\nabla\sigma_0 \cdot \mathbf{n}$ on ∂T . Denote by $\mathcal{W}(T)$ the space of all weak functions on T ; i.e.,

$$(4) \quad \mathcal{W}(T) = \{\sigma = \{\sigma_0, \sigma_b, \sigma_n\} : \sigma_0 \in L^2(T), \sigma_b \in L^2(\partial T), \sigma_n \in L^2(\partial T)\}.$$

For simplicity, denote by $\mathcal{L} = \nabla \cdot (a\nabla)$. The weak \mathcal{L} operator of $\sigma \in \mathcal{W}(T)$, denoted by $\mathcal{L}_w\sigma$, is defined as a linear functional such that

$$(\mathcal{L}_w\sigma, \phi)_T := (\sigma_0, \mathcal{L}\phi)_T - \langle \sigma_b, a\nabla\phi \cdot \mathbf{n} \rangle_{\partial T} + \langle \sigma_n, \phi \rangle_{\partial T},$$

for all ϕ satisfying $\phi \in H^1(\Omega)$ and $a\nabla\phi \in H(\text{div}; T)$.

Denote by $P_r(T)$ the space of polynomials on the element T with degree no more than r . A discrete version of $\mathcal{L}_w\sigma$, denoted by $\mathcal{L}_{w,r,T}\sigma$, is defined as the unique polynomial in $P_r(T)$ satisfying

$$(5) \quad (\mathcal{L}_{w,r,T}\sigma, w)_T = (\sigma_0, \mathcal{L}w)_T - \langle \sigma_b, a\nabla w \cdot \mathbf{n} \rangle_{\partial T} + \langle \sigma_n, w \rangle_{\partial T}, \forall w \in P_r(T),$$

which, using the usual integration by parts, gives

$$(6) \quad (\mathcal{L}_{w,r,T}\sigma, w)_T = (\mathcal{L}\sigma_0, w)_T + \langle \sigma_0 - \sigma_b, a\nabla w \cdot \mathbf{n} \rangle_{\partial T} - \langle a\nabla\sigma_0 \cdot \mathbf{n} - \sigma_n, w \rangle_{\partial T}.$$

3. Primal-Dual Weak Galerkin Scheme

For any given integer $k \geq 2$, let $W_k(T)$ be the local discrete weak function space; i.e.,

$$W_k(T) = \{ \{ \sigma_0, \sigma_b, \sigma_n \} : \sigma_0 \in P_k(T), \sigma_b \in P_k(e), \sigma_n \in P_{k-1}(e), e \subset \partial T \}.$$

Patching $W_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value σ_b on the interior interface \mathcal{E}_h^0 , we obtain a global weak finite element space W_h ; i.e.,

$$W_h = \{ \{ \sigma_0, \sigma_b, \sigma_n \} : \{ \sigma_0, \sigma_b, \sigma_n \}|_T \in W_k(T), \forall T \in \mathcal{T}_h \}.$$

We further introduce the subspace of W_h with homogeneous Dirichlet and Neumann boundary conditions, denoted by W_h^0 ; i.e.,

$$W_h^0 = \{ v \in W_h : \sigma_b = 0 \text{ on } \Gamma_N^c, \sigma_n = 0 \text{ on } \Gamma_D^c \}.$$

In addition, let M_h be the finite element space consisting of piecewise polynomials of degree $k-2$; i.e.,

$$M_h = \{ w : w|_T \in P_{k-2}(T), \forall T \in \mathcal{T}_h \}.$$

For simplicity, for any $\sigma = \{ \sigma_0, \sigma_b, \sigma_n \} \in W_h$, denote by $\mathcal{L}_w\sigma$ the discrete weak operator $\mathcal{L}_{w,k-2,T}\sigma$ computed by using (5) on each element T ; i.e.,

$$(\mathcal{L}_w\sigma)|_T = \mathcal{L}_{w,k-2,T}(\sigma|_T), \quad \forall \sigma \in W_h.$$

For any $\lambda, \sigma \in W_h$, and $u \in M_h$, we introduce the following bilinear forms

$$\begin{aligned} s(\lambda, \sigma) &= \sum_{T \in \mathcal{T}_h} s_T(\lambda, \sigma), \\ b(u, \sigma) &= \sum_{T \in \mathcal{T}_h} (u, \mathcal{L}_w\sigma)_T, \end{aligned}$$

where

$$s_T(\lambda, \sigma) = h_T^{-1} \langle \lambda_0 - \lambda_b, \sigma_0 - \sigma_b \rangle_{\partial T} + h_T \langle a\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n, a\nabla\sigma_0 \cdot \mathbf{n} - \sigma_n \rangle_{\partial T}.$$

The primal-dual weak Galerkin finite element scheme based on the weak formulation (2) for the elliptic Cauchy model problem (1) is described as follows:

Algorithm 3.1 (PDWG Scheme). Find $(u_h; \lambda_h) \in M_h \times W_h^0$ satisfying

$$(7) \quad s(\lambda_h, \sigma) + b(u_h, \sigma) = -(f, \sigma_0) - \langle g_2, \sigma_b \rangle_{\Gamma_N} + \langle g_1, \sigma_n \rangle_{\Gamma_D}, \quad \forall \sigma \in W_h^0,$$

$$(8) \quad b(v, \lambda_h) = 0, \quad \forall v \in M_h.$$

On each element T , denote by Q_0 the L^2 projection operator onto $P_k(T)$; on each edge or face $e \subset \partial T$, denote by Q_b and Q_n the L^2 projection operators onto $P_k(e)$ and $P_{k-1}(e)$, respectively. For any $w \in H^1(\Omega)$, denote by Q_hw the L^2 projection onto the weak finite element space W_h such that on each element T ,

$$Q_hw = \{ Q_0w, Q_bw, Q_n(a\nabla w \cdot \mathbf{n}) \}.$$

Denote by \mathcal{Q}_h^{k-2} the L^2 projection operator onto the space M_h .

Lemma 3.1. [55] *The L^2 projection operators Q_h and \mathcal{Q}_h^{k-2} satisfy the following commuting property:*

$$(9) \quad \mathcal{L}_w(Q_h w) = \mathcal{Q}_h^{k-2}(\mathcal{L}w), \quad \forall w \in H^1(T), \quad a\nabla w \in H(\operatorname{div}; T).$$

4. Existence and Uniqueness

In this section, we shall establish the solution existence and uniqueness of the PDWG scheme (7)-(8).

In the finite element space W_h , we introduce a semi-norm induced from the stabilizer; i.e.,

$$(10) \quad \|\sigma\| = s(\sigma, \sigma)^{\frac{1}{2}}, \quad \forall \sigma \in W_h.$$

We introduce a semi-norm for the finite element space M_h ; i.e.,

$$(11) \quad \|v\|_1 = \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{L}v\|_T^2 + \sum_{e \in \mathcal{E}_h \setminus \Gamma_N^c} h_T \|[[a\nabla v \cdot \mathbf{n}]]\|_e^2 + \sum_{e \in \mathcal{E}_h \setminus \Gamma_D^c} h_T^{-1} \|[[v]]\|_e^2 \right)^{\frac{1}{2}}, \quad \forall v \in M_h,$$

where $[[v]] = v|_{\partial T_1} - v|_{\partial T_2}$ is the jump of v on the interior edge $e = T_1 \cap T_2 \in \mathcal{E}_h^0$ and $[[v]] = v$ on the boundary edge $e \subset \partial\Omega$; the same definition applies to $[[a\nabla v \cdot \mathbf{n}]]$ with \mathbf{n} being the unit outward normal direction to ∂T .

Lemma 4.1. [52] *Let $k \geq 2$ and \mathcal{T}_h be a shape regular partition of the domain Ω specified in [52]. For $0 \leq t \leq 2$, the following estimates hold true:*

$$(12) \quad \sum_{T \in \mathcal{T}_h} h_T^{2t} \|u - \mathcal{Q}_h^{(k-2)}u\|_{t,T}^2 \lesssim h^{2m} \|u\|_m^2, \quad m \in [t, k-1].$$

Lemma 4.2. [37, 47, 49] *Let Ω be an open bounded and connected domain in \mathbb{R}^d ($d = 2, 3$) with Lipschitz continuous domain boundary $\partial\Omega$. Assume that $\Gamma_D \cap \Gamma_N$ is a non-trivial portion of the domain boundary $\partial\Omega$. The solution of the elliptic Cauchy problem (1) (if it exists) is unique.*

Lemma 4.3. *The following inf-sup condition holds true. For any $v \in M_h$, there exists $\rho_v \in W_h^0$ satisfying*

$$(13) \quad b(v, \rho_v) = \|v\|_1^2, \quad \|\rho_v\| \leq \beta \|v\|_1,$$

where $\beta > 0$ is a constant independent of the meshsize h .

Proof. For any $v \in M_h$, we choose $\rho \in W_h^0$ such that on each element T , $\rho = \{h_T^2 \mathcal{L}v, -h_e [[a\nabla v \cdot \mathbf{n}]], h_e^{-1} [[v]]\}$, with boundary value properly adjusted to satisfy the homogeneous boundary data. From the trace inequality (3), triangle inequality, and (11), we have

$$(14) \quad \begin{aligned} & \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\rho_0 - \rho_b\|_{\partial T}^2 \\ &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \|h_T^2 \mathcal{L}v + h_e [[a\nabla v \cdot \mathbf{n}]]\|_{\partial T}^2 \\ &\lesssim \sum_{T \in \mathcal{T}_h} \left(h_T^3 \|\mathcal{L}v\|_{\partial T}^2 + h_T \|[[a\nabla v \cdot \mathbf{n}]]\|_{\partial T}^2 \right) \\ &\lesssim \sum_{T \in \mathcal{T}_h} \left(h_T^2 \|\mathcal{L}v\|_T^2 + h_T \|[[a\nabla v \cdot \mathbf{n}]]\|_{\partial T}^2 \right) \\ &\lesssim \|v\|_1^2. \end{aligned}$$

Analogously, using the trace inequality (3), inverse inequality, triangle inequality, and (11) gives

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_h} h_T \|a \nabla \rho_0 \cdot \mathbf{n} - [[v]]\|_{\partial T}^2 \\
&= \sum_{T \in \mathcal{T}_h} h_T \|a \nabla (h_T^2 \mathcal{L}v) \cdot \mathbf{n} - h_T^{-1} [[v]]\|_{\partial T}^2 \\
(15) \quad & \lesssim \sum_{T \in \mathcal{T}_h} h_T \left(h_T^4 \|a \nabla (\mathcal{L}v) \cdot \mathbf{n}\|_{\partial T}^2 + h_T^{-2} \|[[v]]\|_{\partial T}^2 \right) \\
& \lesssim \sum_{T \in \mathcal{T}_h} \left(h_T^2 \|\mathcal{L}v\|_T^2 + h_T^{-1} \|[[v]]\|_{\partial T}^2 \right) \\
& \lesssim \|v\|_1^2.
\end{aligned}$$

From (10)-(11), combining the estimates (14)-(15) yields

$$(16) \quad \|\rho\| \lesssim \|v\|_1.$$

Next, using (5) yields

$$\begin{aligned}
b(v, \rho) &= \sum_{T \in \mathcal{T}_h} (v, \mathcal{L}_w \rho)_T \\
&= \sum_{T \in \mathcal{T}_h} (\rho_0, \mathcal{L}v)_T - \langle \rho_b, a \nabla v \cdot \mathbf{n} \rangle_{\partial T} + \langle \rho_n, v \rangle_{\partial T} \\
(17) \quad &= \sum_{T \in \mathcal{T}_h} (\rho_0, \mathcal{L}v)_T - \sum_{e \in \mathcal{E}_h} \langle \rho_b, [[a \nabla v \cdot \mathbf{n}]] \rangle_e + \sum_{e \in \mathcal{E}_h} \langle \rho_n, [[v]] \rangle_e \\
&= \sum_{T \in \mathcal{T}_h} (\rho_0, \mathcal{L}v)_T - \sum_{e \in \mathcal{E}_h \setminus \Gamma_N^c} \langle \rho_b, [[a \nabla v \cdot \mathbf{n}]] \rangle_e + \sum_{e \in \mathcal{E}_h \setminus \Gamma_D^c} \langle \rho_n, [[v]] \rangle_e \\
&= \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{L}v\|_T^2 + \sum_{e \in \mathcal{E}_h \setminus \Gamma_N^c} h_T \|[[a \nabla v \cdot \mathbf{n}]]\|_e^2 + \sum_{e \in \mathcal{E}_h \setminus \Gamma_D^c} h_T^{-1} \|[[v]]\|_e^2 \\
&= \|v\|_1^2,
\end{aligned}$$

which, together with (16), completes the proof of the lemma. \square

The following theorem is the main result on solution existence and uniqueness for the PDWG scheme (7)-(8).

Theorem 4.4. *Assume that $\Gamma_D \cap \Gamma_N$ contains a nontrivial portion of the domain boundary $\partial\Omega$ and $\Gamma_D \cup \Gamma_N \Subset \partial\Omega$ is a proper closed subset. The PDWG finite element algorithm (7)-(8) has one and only one solution.*

Proof. It suffices to show that zero is the unique solution to the problem (7)-(8) with homogeneous data $f = 0$, $g_1 = 0$ and $g_2 = 0$. To this end, assume $f = 0$, $g_1 = 0$ and $g_2 = 0$ in (7)-(8). By letting $v = u_h$ and $\sigma = \lambda_h$, the difference of (8) and (7) gives $s(\lambda_h, \lambda_h) = 0$, which implies $\lambda_0 = \lambda_b$ and $a \nabla \lambda_0 \cdot \mathbf{n} = \lambda_n$ on each ∂T . Therefore, together with the fact that $\lambda_h \in W_h^0$, we obtain $\lambda_0 = 0$ on Γ_N^c and $a \nabla \lambda_0 \cdot \mathbf{n} = 0$ on Γ_D^c .

It follows from (8), (6) and the usual integration by parts that for any $v \in M_h$

$$\begin{aligned}
 (18) \quad & 0 = b(v, \lambda_h) \\
 & = \sum_{T \in \mathcal{T}_h} (v, \mathcal{L}_w \lambda_h)_T \\
 & = \sum_{T \in \mathcal{T}_h} (\mathcal{L} \lambda_0, v)_T + \langle \lambda_0 - \lambda_b, a \nabla v \cdot \mathbf{n} \rangle_{\partial T} - \langle a \nabla \lambda_0 \cdot \mathbf{n} - \lambda_n, v \rangle_{\partial T} \\
 & = \sum_{T \in \mathcal{T}_h} (\mathcal{L} \lambda_0, v)_T,
 \end{aligned}$$

where we have used $\lambda_0 = \lambda_b$ and $a \nabla \lambda_0 \cdot \mathbf{n} = \lambda_n$ on each ∂T . This implies $\mathcal{L} \lambda_0 = 0$ on each element $T \in \mathcal{T}_h$ by taking $v = \mathcal{L} \lambda_0$, which further yields $\mathcal{L} \lambda_0 = 0$ in Ω . Since $\Gamma_D \cup \Gamma_N \Subset \partial \Omega$ is a proper closed subset, $\Gamma_D^c \cap \Gamma_N^c = (\Gamma_D \cup \Gamma_N)^c$ contains a nontrivial portion of $\partial \Omega$. Since $\lambda_0 = 0$ on Γ_N^c and $a \nabla \lambda_0 \cdot \mathbf{n} = 0$ on Γ_D^c , from Lemma 4.2, we get $\lambda_0 \equiv 0$ in Ω . It follows that $\lambda_h \equiv 0$ in Ω , as $\lambda_b = \lambda_0$ and $\lambda_n = a \nabla \lambda_0 \cdot \mathbf{n}$ on each ∂T .

To demonstrate $u_h \equiv 0$, we use $\lambda_h \equiv 0$ in Ω and the equation (7) to obtain

$$(19) \quad b(u_h, \sigma) = 0, \quad \forall \sigma \in W_h^0,$$

which, together with Lemma 4.3, leads to $\|u_h\|_1 = 0$. This yields $\mathcal{L} u_h = 0$ on each element $T \in \mathcal{T}_h$, $[[a \nabla u_h \cdot \mathbf{n}]] = 0$ on each $e \in \mathcal{E}_h^0$ and $[[u_h]] = 0$ on each $e \in \mathcal{E}_h^0$, and $a \nabla u_h \cdot \mathbf{n} = 0$ on each $e \subset \partial \Omega \setminus \Gamma_N^c$ and $u_h = 0$ on each $e \subset \partial \Omega \setminus \Gamma_D^c$. Therefore, we have $\mathcal{L} u_h = 0$ in Ω , $a \nabla u_h \cdot \mathbf{n} = 0$ on $\partial \Omega \setminus \Gamma_N^c$ and $u_h = 0$ on $\partial \Omega \setminus \Gamma_D^c$, for which there exists a unique solution $u_h = 0$ in Ω , provided that $(\partial \Omega \setminus \Gamma_N^c) \cap (\partial \Omega \setminus \Gamma_D^c) = \Gamma_N \cap \Gamma_D$ is a nontrivial portion of the domain boundary $\partial \Omega$.

This completes the proof of the theorem. \square

5. Error Equations

This section is devoted to deriving the error equations for the PDWG scheme (7)-(8) which will play a critical role in establishing the error estimates in the following section.

Let u and $(u_h, \lambda_h) \in M_h \times W_h^0$ be the exact solution of the elliptic Cauchy model problem (1) and PDWG solution of the numerical scheme (7)-(8), respectively. Note that the Lagrange multiplier λ_h approximates the trivial solution $\lambda = 0$. The error functions are defined as the difference between the numerical solution (u_h, λ_h) and the L^2 projection of the exact solution of (1); i.e.,

$$(20) \quad e_h = u_h - \mathcal{Q}_h^{k-2} u,$$

$$(21) \quad \varepsilon_h = \lambda_h - Q_h \lambda = \lambda_h.$$

Lemma 5.1. *For any $\sigma \in W_h$ and $v \in M_h$, the following identity holds true:*

$$(\mathcal{L}_w \sigma, v)_T = (\mathcal{L} \sigma_0, v)_T + R_T(\sigma, v),$$

where

$$R_T(\sigma, v) = \langle \sigma_0 - \sigma_b, a \nabla v \cdot \mathbf{n} \rangle_{\partial T} - \langle a \nabla \sigma_0 \cdot \mathbf{n} - \sigma_n, v \rangle_{\partial T}.$$

Proof. This proof can be easily obtained using (6). \square

Lemma 5.2. *Let u and $(u_h, \lambda_h) \in M_h \times W_h^0$ be the solutions arising from (1) and (7)-(8), respectively. The error functions e_h and ε_h satisfy the following error*

equations; i.e.,

$$(22) \quad s(\varepsilon_h, \sigma) + b(e_h, \sigma) = \ell_u(\sigma), \quad \forall \sigma \in W_h^0,$$

$$(23) \quad b(v, \varepsilon_h) = 0, \quad \forall v \in M_h,$$

where $\ell_u(\sigma)$ is given by

$$(24) \quad \ell_u(\sigma) = \sum_{T \in \mathcal{T}_h} \langle \sigma_0 - \sigma_b, a \nabla(u - \mathcal{Q}_h^{k-2}u) \cdot \mathbf{n} \rangle_{\partial T} - \langle a \nabla \sigma_0 \cdot \mathbf{n} - \sigma_n, u - \mathcal{Q}_h^{k-2}u \rangle_{\partial T}.$$

Proof. From (21) and (8) we have

$$b(v, \varepsilon_h) = b(v, \lambda_h) = 0, \quad \forall v \in M_h,$$

which gives rise to (23).

Recall that $\lambda = 0$. From (7) we arrive at

$$(25) \quad \begin{aligned} & s(\lambda_h - \mathcal{Q}_h \lambda, \sigma) + b(u_h - \mathcal{Q}_h^{k-2}u, \sigma) \\ &= - \langle f, \sigma_0 \rangle - \langle g_2, \sigma_b \rangle_{\Gamma_N} + \langle g_1, \sigma_n \rangle_{\Gamma_D} - b(\mathcal{Q}_h^{k-2}u, \sigma). \end{aligned}$$

As to the term $b(\mathcal{Q}_h^{k-2}u, \sigma)$, using Lemma 5.1 and the usual integration by parts gives

$$(26) \quad \begin{aligned} & b(\mathcal{Q}_h^{k-2}u, \sigma) \\ &= \sum_{T \in \mathcal{T}_h} (\mathcal{Q}_h^{k-2}u, \mathcal{L}_w \sigma)_T \\ &= \sum_{T \in \mathcal{T}_h} (\mathcal{L} \sigma_0, \mathcal{Q}_h^{k-2}u)_T + R_T(\sigma, \mathcal{Q}_h^{k-2}u) \\ &= \sum_{T \in \mathcal{T}_h} (\mathcal{L} \sigma_0, u)_T + R_T(\sigma, \mathcal{Q}_h^{k-2}u) \\ &= \sum_{T \in \mathcal{T}_h} (\sigma_0, \nabla \cdot (a \nabla u))_T - \langle \sigma_0, a \nabla u \cdot \mathbf{n} \rangle_{\partial T} + \langle a \nabla \sigma_0 \cdot \mathbf{n}, u \rangle_{\partial T} + R_T(\sigma, \mathcal{Q}_h^{k-2}u). \end{aligned}$$

Since u is the exact solution of (1), $\sigma_b = 0$ on Γ_N^c and $\sigma_n = 0$ on Γ_D^c , we have

$$(27) \quad \sum_{T \in \mathcal{T}_h} \langle \sigma_b, a \nabla u \cdot \mathbf{n} \rangle_{\partial T} = \langle \sigma_b, g_2 \rangle_{\Gamma_N},$$

$$(28) \quad \sum_{T \in \mathcal{T}_h} \langle \sigma_n, u \rangle_{\partial T} = \langle \sigma_n, g_1 \rangle_{\Gamma_D}.$$

Substituting (27), (28) and (1) into (26), we arrive at

$$(29) \quad \begin{aligned} & b(\mathcal{Q}_h^{k-2}u, \sigma) \\ &= - \langle \sigma_0, f \rangle - \sum_{T \in \mathcal{T}_h} \langle \sigma_0 - \sigma_b, a \nabla u \cdot \mathbf{n} \rangle_{\partial T} + \langle a \nabla \sigma_0 \cdot \mathbf{n} - \sigma_n, u \rangle_{\partial T} \\ & \quad - \langle \sigma_b, g_2 \rangle_{\Gamma_N} + \langle \sigma_n, g_1 \rangle_{\Gamma_D} + R_T(\sigma, \mathcal{Q}_h^{k-2}u). \end{aligned}$$

Substituting (29) into (25) gives rise to the error equation (22), which completes the proof of the lemma. \square

6. Error Estimates

In this section, we shall demonstrate the optimal order of error estimates for the PDWG scheme (7)-(8).

Theorem 6.1. *Let u be the exact solution of the elliptic Cauchy model problem (1) and $(u_h, \lambda_h) \in M_h \times W_h^0$ be the numerical solution arising from PDWG method (7)-(8) with $k \geq 2$. Assume (1) the coefficient tensor a is piecewise constants in Ω with respect to the finite element partition \mathcal{T}_h which is shape regular as specified in [52]; (2) the exact solution u is sufficiently regular such that $u \in H^m(\Omega)$ ($2 - \theta \leq m \leq k - 1$) with $0 \leq \theta < \frac{1}{2}$. The following error estimates hold true; i.e.,*

$$(30) \quad \|\lambda_h\| + \|e_h\|_1 \lesssim h^{m-1} \|u\|_m.$$

Proof. Letting $\sigma = \varepsilon_h = \{\varepsilon_0, \varepsilon_b, \varepsilon_n\}$ in the error equation (22) and using (23) and (24) we arrive at

$$(31) \quad \begin{aligned} s(\varepsilon_h, \varepsilon_h) &= \ell_u(\varepsilon_h) \\ &= \sum_{T \in \mathcal{T}_h} \langle \varepsilon_0 - \varepsilon_b, a \nabla(u - \mathcal{Q}_h^{k-2}u) \cdot \mathbf{n} \rangle_{\partial T} + \langle \varepsilon_n - a \nabla \varepsilon_0 \cdot \mathbf{n}, u - \mathcal{Q}_h^{k-2}u \rangle_{\partial T}. \end{aligned}$$

Using Cauchy-Schwarz inequality, the trace inequality (3), the estimate (12) and (10) gives

$$(32) \quad \begin{aligned} &\left| \sum_{T \in \mathcal{T}_h} \langle \varepsilon_0 - \varepsilon_b, a \nabla(u - \mathcal{Q}_h^{k-2}u) \cdot \mathbf{n} \rangle_{\partial T} \right| \\ &\lesssim \left(\sum_{T \in \mathcal{T}_h} \|\varepsilon_0 - \varepsilon_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|a \nabla(u - \mathcal{Q}_h^{k-2}u)\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|u - \mathcal{Q}_h^{k-2}u\|_{1,T}^2 + h_T^{1-2\theta} \|u - \mathcal{Q}_h^{k-2}u\|_{2-\theta,T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\varepsilon_0 - \varepsilon_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\lesssim h^{m-1} \|u\|_m \|\varepsilon_h\|. \end{aligned}$$

Similarly, from the Cauchy-Schwarz inequality and the trace inequality (3), the estimate (12) and (10), we obtain

$$(33) \quad \begin{aligned} &\left| \sum_{T \in \mathcal{T}_h} \langle \varepsilon_n - a \nabla \varepsilon_0 \cdot \mathbf{n}, u - \mathcal{Q}_h^{k-2}u \rangle_{\partial T} \right| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} \|\varepsilon_n - a \nabla \varepsilon_0 \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|u - \mathcal{Q}_h^{k-2}u\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{T \in \mathcal{T}_h} \|\varepsilon_n - a \nabla \varepsilon_0 \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|u - \mathcal{Q}_h^{k-2}u\|_T^2 + h_T^{1-2\theta} \|\nabla(u - \mathcal{Q}_h^{k-2}u)\|_{1-\theta,T}^2 \right)^{\frac{1}{2}} \\ &\lesssim h^{m-1} \|u\|_m \|\varepsilon_h\|. \end{aligned}$$

Substituting the estimates (32) and (33) into (31) yields

$$(34) \quad \|\varepsilon_h\|^2 = |\ell_u(\varepsilon_h)| \lesssim h^{m-1} \|u\|_m \|\varepsilon_h\|,$$

which leads to

$$(35) \quad \|\varepsilon_h\| \lesssim h^{m-1} \|u\|_m.$$

Furthermore, the error equation (22) yields

$$b(e_h, \sigma) = \ell_u(\sigma) - s(\varepsilon_h, \sigma), \quad \forall \sigma \in W_h^0.$$

It follows from (34)-(35), Cauchy-Schwarz inequality, triangle inequality, and (10) that

$$|b(e_h, \sigma)| \leq |\ell_u(\sigma)| + \|\varepsilon_h\| \|\sigma\| \lesssim h^{m-1} \|u\|_m \|\sigma\|$$

for all $\sigma \in W_h^0$. Thus, from the *inf-sup* condition (13) we obtain

$$\|e_h\|_1 \lesssim h^{m-1} \|u\|_m,$$

which, together with the error estimate (35), completes the proof of the theorem. \square

For C^0 -WG elements (i.e., $\sigma_b = \sigma_0$), $\ell_u(\sigma)$ in (24) is simplified into

$$(36) \quad \ell_u(\sigma) = - \sum_{T \in \mathcal{T}_h} \langle a \nabla \sigma_0 \cdot \mathbf{n} - \sigma_n, u - \mathcal{Q}_h^{k-2} u \rangle_{\partial T},$$

which is applicable to derive the error estimate for the primal variable under ultra-low regularity assumption $u \in H^{1-\theta}(\Omega)$ for $0 \leq \theta < \frac{1}{2}$. The conclusion is stated in the following corollary.

Corollary 6.2. *Let u be the exact solution of the elliptic Cauchy model problem (1) and $(u_h, \lambda_h) \in M_h \times W_h^0$ be the numerical solution arising from PDWG method (7)-(8) with $k \geq 2$. Assume (1) the coefficient tensor a is piecewise constants in Ω with respect to the finite element partition \mathcal{T}_h which is shape regular as specified in [52]; (2) the exact solution u is sufficiently regular such that $u \in H^m(\Omega)$ ($1 - \theta \leq m \leq k - 1$) with $0 \leq \theta < \frac{1}{2}$. The following error estimates hold true; i.e.,*

$$\|\lambda_h\| + \|e_h\|_1 \lesssim h^{m-1} \|u\|_m.$$

Proof. This proof is easily obtained by following the proof of Theorem 6.1 and (36). \square

7. Numerical Experiments

Extensive numerical results for the PDWG finite element scheme (7)-(8) are reported in this section. The finite element partition \mathcal{T}_h is generated through a successive uniform refinement for a coarse triangulation of the unit square domain $\Omega = (0, 1)^2$ by dividing each coarse level triangular element into four congruent sub-triangles by connecting the three mid-points on the edges. The numerical tests are based on the lowest order $k = 2$, where the finite element space for the primal variable is $M_{h,2} = \{u_h : u_h|_T \in P_0(T), \forall T \in \mathcal{T}_h\}$, and the weak finite element space for the dual variable is $W_{h,2} = \{\lambda_h = \{\lambda_0, \lambda_b, \lambda_n\} : \lambda_0 \in P_2(T), \lambda_b \in P_2(e), \lambda_n \in P_1(e), e \subset \partial T, \forall T \in \mathcal{T}_h\}$.

Let $u_h \in M_{h,2}$ and $\lambda_h = \{\lambda_0, \lambda_b, \lambda_n\} \in W_{h,2}$ be the numerical solutions arising from the PDWG scheme (7)-(8). The primal variable u_h is compared with the exact solution u at the center of each element which is known as the nodal point interpolation $I_h u$. The auxiliary variable λ_h approximates the true solution $\lambda = 0$, and is compared with $Q_h \lambda = 0$. The error functions are respectively denoted by

$$\varepsilon_h = \lambda_h - Q_h \lambda = \{\lambda_0, \lambda_b, \lambda_n\}, \quad e_h = u_h - I_h u.$$

TABLE 1. Numerical error with exact solution $u = 1$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary edge $(0, 1) \times 0$.

$1/h$	$\ e_h\ _0$	$\ \lambda_0\ _0$	$\ \lambda_b\ _0$	$\ \lambda_n\ _0$
1	0E-16	0.03131E-16	0.05373E-16	0.1412E-16
2	0.05537E-14	0.01277E-14	0.01931E-14	0.1119E-14
4	0.2243E-14	0.02680E-14	0.04223E-14	0.2141E-14
8	0.5730E-13	0.03449E-13	0.05200E-13	0.1962E-13
16	0.1824E-12	0.009643E-12	0.01422E-12	0.06788E-12
32	0.1406E-11	0.001204E-11	0.001745E-11	0.01216E-11

TABLE 2. Numerical error with exact solution $u = 1$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary edge $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	$\ \lambda_0\ _0$	$\ \lambda_b\ _0$	$\ \lambda_n\ _0$
1	0.2220E-15	0.07805E-15	0.1246E-15	0.5440E-15
2	0.1407E-14	0.03347E-14	0.05395E-14	0.1939E-14
4	0.2757E-14	0.01359E-14	0.01797E-14	0.2002E-14
8	0.2490E-13	0.009452E-13	0.01398E-13	0.1041E-13
16	0.1257E-12	0.001696E-12	0.002536E-12	0.02520E-12
32	0.8767E-12	0.01485E-12	0.02134E-12	0.09533E-12

TABLE 3. Numerical convergence with exact solution $u = \sin(x)\sin(y)$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$ and Neumann on the boundary segments $0 \times (0, 1)$ and $1 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
2	0.02025		0.07907		0.1255		0.7059	
4	0.008189	1.306	0.01649	2.261	0.02475	2.343	0.1705	2.050
8	0.003478	1.235	0.003921	2.073	0.005726	2.112	0.04248	2.005
16	0.001604	1.117	0.0009725	2.012	0.001399	2.033	0.01063	1.999
32	0.0007741	1.051	0.0002434	1.998	0.0003473	2.010	0.002659	1.999

The following L^2 norms are used to measure the error functions; i.e.,

$$\|e_h\|_0 = \left(\sum_{T \in \mathcal{T}_h} \int_T e_h^2 dT \right)^{\frac{1}{2}}, \quad \|\lambda_0\|_0 = \left(\sum_{T \in \mathcal{T}_h} \int_T \lambda_0^2 dT \right)^{\frac{1}{2}},$$

$$\|\lambda_b\|_0 = \left(\sum_{T \in \mathcal{T}_h} h_T \int_{\partial T} \lambda_b^2 ds \right)^{\frac{1}{2}}, \quad \|\lambda_n\|_0 = \left(\sum_{T \in \mathcal{T}_h} h_T \int_{\partial T} \lambda_n^2 ds \right)^{\frac{1}{2}}.$$

Tables 1-2 indicate the correctness and reliability of the code for the elliptic Cauchy model problem (1). The configuration of the test case is as follows: (1) the exact solution is $u = 1$; (2) the coefficient tensor is $a = [1, 0; 0, 1]$; (3) both the Dirichlet and Neumann boundary conditions are set on the horizontal boundary segment $(0, 1) \times 0$ (see Table 1) or on the vertical boundary segment $0 \times (0, 1)$ (see

TABLE 4. Numerical convergence with exact solution $u = \sin(\pi x) \cos(\pi y)$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$ and Neumann on the boundary segments $0 \times (0, 1)$ and $1 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.1983		1.775		3.035		5.942	
2	0.07463	1.410	0.2382	2.897	0.3974	2.933	1.147	2.373
4	0.02809	1.410	0.04732	2.332	0.07001	2.505	0.3657	1.650
8	0.01183	1.247	0.01125	2.072	0.01625	2.107	0.09009	2.021
16	0.004935	1.262	0.002793	2.010	0.003997	2.024	0.02209	2.028
32	0.002181	1.178	0.0006993	1.998	0.0009954	2.006	0.005478	2.012

TABLE 5. Numerical convergence with exact solution $u = xy(1 - x)(1 - y)$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$ and Neumann on the boundary segments $0 \times (0, 1)$ and $1 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.03376		0.1206		0.2025		0.1335	
2	0.008454	1.998	0.02765	2.125	0.03792	2.417	0.04527	1.560
4	0.002336	1.856	0.006620	2.062	0.009235	2.038	0.01260	1.845
8	0.0008282	1.496	0.001651	2.003	0.002325	1.990	0.003207	1.975
16	0.0003441	1.267	0.0004133	1.998	0.0005838	1.994	0.0007990	2.005
32	0.0001560	1.142	0.0001034	1.999	0.0001462	1.998	0.0001979	2.014

TABLE 6. Numerical convergence with exact solution $u = e^{xy}$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$ and Neumann on the boundary segments $0 \times (0, 1)$ and $1 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
2	0.05293		0.1790		0.2650		1.604	
4	0.02334	1.181	0.04208	2.089	0.06047	2.132	0.4208	1.931
8	0.009735	1.262	0.01057	1.993	0.01506	2.005	0.1080	1.962
16	0.004345	1.164	0.002679	1.980	0.003805	1.985	0.02742	1.978
32	0.002050	1.084	0.0006762	1.986	0.0009587	1.989	0.006912	1.988

Table 2). Note that the numerical solution is consistent with the exact solution $u = 1$. We observe from Tables 1-2 that the errors are in machine accuracy, especially for relatively coarse grids, which perfectly consists with the developed theory. However, the error seems to deteriorate for the finer meshes which may be caused by the ill-posedness of the elliptic Cauchy problem and/or the poor conditioning of the discrete linear system.

Tables 3-6 demonstrate the numerical performance of the PDWG scheme (7)-(8) when the exact solutions are given by $u = \sin(x) \sin(y)$, $u = \sin(\pi x) \cos(\pi y)$, $u = xy(1 - x)(1 - y)$ and $u = e^{xy}$, respectively. The coefficient tensor is $a =$

TABLE 7. Numerical convergence with exact solution $u = \sin(x)\sin(y)$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Dirichlet on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.07102		0.3979		0.6416		2.777	
2	0.03856	0.8810	0.06642	2.583	0.1032	2.636	0.4885	2.507
4	0.01390	1.473	0.02039	1.703	0.03087	1.741	0.1443	1.759
8	0.003625	1.939	0.006680	1.610	0.009868	1.645	0.04291	1.750
16	0.001311	1.467	0.002131	1.648	0.003087	1.676	0.01235	1.796
32	0.0004754	1.463	0.0007006	1.605	0.001005	1.620	0.004066	1.603

TABLE 8. Numerical convergence with exact solution $u = \cos(x)\cos(y)$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Dirichlet on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
2	0.02161		0.09323		0.138		0.4247	
4	0.009932	1.121	0.02531	1.881	0.03693	1.904	0.1116	1.928
8	0.003352	1.567	0.006692	1.919	0.009652	1.936	0.02920	1.934
16	0.001396	1.264	0.001836	1.866	0.002630	1.876	0.007239	2.012
32	0.0006223	1.165	0.0005124	1.841	0.0007307	1.847	0.001986	1.866

TABLE 9. Numerical convergence with exact solution $u = xy(1 - x)(1 - y)$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Dirichlet on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.01042		0.07688		0.1406		0.2900	
2	0.004678	1.155	0.02185	1.815	0.02976	2.240	0.03095	3.228
4	0.001630	1.521	0.005182	2.076	0.007245	2.038	0.005847	2.404
8	0.0005178	1.655	0.001261	2.039	0.001780	2.025	0.001644	1.831
16	0.0002162	1.260	0.0003168	1.993	0.0004484	1.989	0.0004812	1.772
32	9.508E-05	1.185	8.028E-05	1.980	0.0001137	1.979	0.0001483	1.698

$[1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$. The boundary conditions are set as follows: (1) Dirichlet on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$; and (2) Neumann on the boundary segments $0 \times (0, 1)$ and $1 \times (0, 1)$. Note that this is a standard mixed boundary value problem without Cauchy data given on the boundary. The numerical results show that the convergence rate for e_h in the discrete L^2 norm is of an order a bit higher than $\mathcal{O}(h)$. This group of numerical examples imply the effectiveness of the PDWG algorithm (7)-(8) for the classical well-posed problems.

TABLE 10. Numerical convergence with exact solution $u = e^{xy}$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Dirichlet on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.1067		0.6226		0.8525		4.572	
2	0.05165	1.047	0.1483	2.070	0.2128	2.002	0.9598	2.252
4	0.01646	1.650	0.04342	1.772	0.06371	1.740	0.2797	1.779
8	0.005303	1.634	0.01198	1.858	0.01749	1.865	0.07875	1.828
16	0.002344	1.178	0.002741	2.128	0.003944	2.149	0.01915	2.040
32	0.0009561	1.294	0.0007732	1.826	0.001105	1.836	0.005242	1.869

TABLE 11. Numerical convergence with exact solution $u = \sin(\pi x) \sin(\pi y)$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.1281		0.8807		1.501		0.3968	
2	0.04934	1.377	0.2939	1.583	0.4002	1.907	0.3624	0.1307
4	0.02366	1.060	0.07421	1.986	0.1037	1.949	0.08287	2.129
8	0.01312	0.8503	0.01867	1.991	0.02633	1.977	0.02670	1.634
16	0.006379	1.041	0.004740	1.977	0.006706	1.973	0.008679	1.621
32	0.003017	1.080	0.001215	1.964	0.001720	1.963	0.002968	1.548

TABLE 12. Numerical convergence with exact solution $u = xy(1-x)(1-y)$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
2	0.002822		0.02069		0.02818		0.02300	
4	0.001610	0.8103	0.005023	2.042	0.007021	2.005	0.005487	2.068
8	0.0008414	0.9358	0.001246	2.011	0.001758	1.998	0.001748	1.650
16	0.0003811	1.142	0.0003160	1.979	0.0004471	1.975	0.0005608	1.640
32	0.0001708	1.158	8.086E-05	1.967	0.0001145	1.966	0.0001848	1.602

Tables 7-10 demonstrate the performance of the PDWG algorithm (7)-(8) when the exact solutions are given by $u = \sin(x) \sin(y)$, $u = \cos(x) \cos(y)$, $u = xy(1-x)(1-y)$ and $u = e^{xy}$, respectively. The setting of boundary conditions is as follows: (1) both Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$; and (2) Dirichlet on the boundary segment $0 \times (0, 1)$. The coefficient tensor is $a = [1, 0; 0, 1]$. The numerical results in Tables 7-10 show that the convergence order for the primal variable is higher than $\mathcal{O}(h)$.

Tables 11-12 illustrate the numerical performance of the PDWG algorithm when both Dirichlet and Neumann conditions are given on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$. The exact solutions are taken by $u = \sin(\pi x) \sin(\pi y)$ and $u = xy(1-x)(1-y)$, respectively; and the coefficient tensor is $a = [1, 0; 0, 1]$. We can

TABLE 13. Numerical convergence with exact solution $u = e^{xy}$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
2	0.09498		0.1600		0.2125		1.629	
4	0.08063	0.2362	0.03763	2.088	0.05258	2.015	0.4707	1.791
8	0.03599	1.164	0.01069	1.821	0.01550	1.762	0.1461	1.688
16	0.01465	1.297	0.002853	1.900	0.004152	1.900	0.04208	1.796
32	0.006831	1.100	0.0008062	1.823	0.001161	1.838	0.01166	1.852

TABLE 14. Numerical convergence with exact solution $u = \sin(x) \cos(y)$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Neumann on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.05811		1.050		1.993		0.8839	
2	0.02234	1.379	0.2415	2.121	0.4034	2.304	0.1804	2.293
4	0.009729	1.199	0.06850	1.818	0.1056	1.934	0.05650	1.675
8	0.004034	1.270	0.02069	1.727	0.03058	1.788	0.02364	1.257
16	0.001292	1.642	0.005941	1.800	0.008589	1.832	0.008732	1.437
32	0.0005206	1.311	0.001395	2.090	0.001994	2.107	0.002616	1.739

TABLE 15. Numerical convergence with exact solution $u = e^{xy}$; the coefficient tensor $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Neumann on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.1009		2.097		3.548		3.380	
2	0.05792	0.8002	0.4685	2.162	0.7333	2.275	1.300	1.379
4	0.02899	0.9985	0.1087	2.108	0.1647	2.154	0.5187	1.326
8	0.01315	1.140	0.01971	2.463	0.02922	2.495	0.1603	1.694
16	0.004980	1.401	0.005810	1.762	0.008517	1.779	0.04668	1.780
32	0.002143	1.216	0.001291	2.170	0.001858	2.196	0.01099	2.086

observe from the numerical results that the convergence rate for e_h in the discrete L^2 norm is of an order a bit higher than $\mathcal{O}(h)$.

Table 13 presents the numerical performance of the PDWG algorithm in the following configuration: (1) the exact solution is $u = e^{xy}$; (2) the coefficient tensor is $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; (3) both Dirichlet and Neumann conditions are given on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$. We can observe from Table 13 that the convergence rate for the numerical solution u_h in the discrete L^2 norm is of an order a little higher than $\mathcal{O}(h)$.

Tables 14-15 demonstrate the performance of the PDWG scheme for the numerical tests in the following configuration: (1) the coefficient tensor is $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$; (2) the boundary conditions are set by Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$ and Neumann on the

TABLE 16. Numerical convergence with exact solution $u = xy(1-x)(1-y)$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Dirichlet on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.01042		0.07688		0.1406		0.2900	
2	0.004678	1.155	0.02185	1.815	0.02976	2.240	0.03095	3.228
4	0.001630	1.521	0.005182	2.076	0.007245	2.038	0.005847	2.404
8	0.0005178	1.655	0.001261	2.040	0.001780	2.025	0.001644	1.831
16	0.0002162	1.260	0.0003168	1.993	0.0004484	1.989	0.0004812	1.772
32	9.508E-05	1.185	8.028E-05	1.980	0.0001137	1.980	0.0001483	1.698

TABLE 17. Numerical convergence with exact solution $u = e^{xy}$; the coefficient tensor $a = [1, 0; 0, 1]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and Dirichlet on the boundary segment $0 \times (0, 1)$.

$1/h$	$\ e_h\ _0$	order	$\ \lambda_0\ _0$	order	$\ \lambda_b\ _0$	order	$\ \lambda_n\ _0$	order
1	0.1067		0.6226		0.8525		4.572	
2	0.05165	1.047	0.1483	2.070	0.2128	2.002	0.9598	2.252
4	0.01646	1.650	0.04342	1.772	0.06371	1.740	0.27965	1.779
8	0.005303	1.634	0.01197662	1.858	0.01749	1.865	0.07875	1.828
16	0.002344	1.178	0.002741	2.128	0.003944	2.149	0.01915	2.040
32	0.0009561	1.294	0.0007732	1.826	0.001105	1.836	0.005242	1.869

TABLE 18. Numerical convergence with exact solutions $u_1 = \sin(x) \cos(y)$, $u_2 = \sin(\pi x) \sin(\pi y)$, $u_3 = \sin(x) \sin(y)$ and $u_4 = e^{xy}$; the coefficient tensor $a = [1+x^2, 0.25xy; 0.25xy, 1+y^2]$; Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, Dirichlet on the boundary segment $0 \times (0, 1)$ and Neumann on the boundary segment $1 \times (0, 1)$.

$1/h$	$\ e_h\ _0(u_1)$	order	$\ e_h\ _0(u_2)$	order	$\ e_h\ _0(u_3)$	order	$\ e_h\ _0(u_4)$	order
2	0.01916		0.06362		0.02025		0.06538	
4	0.006850	1.484	0.02385	1.415	0.008189	1.306	0.02519	1.376
8	0.002455	1.480	0.008741	1.448	0.003478	1.235	0.01014	1.312
16	0.0008979	1.451	0.003645	1.262	0.001604	1.117	0.004450	1.189
32	0.0003603	1.317	0.001663	1.132	0.0007741	1.051	0.002072	1.103

boundary segment $0 \times (0, 1)$; (3) the exact solutions are given by $u = \sin(x) \cos(y)$ and $u = e^{xy}$, respectively. Tables 14-15 indicate that the convergence order for the PDWG solution u_h in the discrete L^2 norm is of an order higher than the order $\mathcal{O}(h)$.

Tables 16-17 demonstrate the performance of the PDWG algorithm for the exact solutions $u = xy(1-x)(1-y)$ and $u = e^{xy}$, respectively. The boundary conditions

are set as follows: (1) Dirichlet and Neumann on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$, and (2) Neumann on the boundary segment $0 \times (0, 1)$. The coefficient tensor is $a = [1, 0; 0, 1]$. The numerical results in Tables 16-17 illustrate that the convergence rate for e_h in the discrete L^2 norm is higher than the order $\mathcal{O}(h)$.

Table 18 demonstrate the performance of the PDWG method when the exact solutions are given by $u_1 = \sin(x) \cos(y)$, $u_2 = \sin(\pi x) \sin(\pi y)$, $u_3 = \sin(x) \sin(y)$ and $u_4 = e^{xy}$ respectively. The boundary conditions are set as follows: (1) Dirichlet and Neumann conditions on the boundary segments $(0, 1) \times 0$ and $(0, 1) \times 1$; (2) Dirichlet on the boundary segment $0 \times (0, 1)$; and (3) Neumann on the boundary segment $1 \times (0, 1)$. The coefficient tensor is chosen by $a = [1 + x^2, 0.25xy; 0.25xy, 1 + y^2]$. We observe from Table 18 that the convergence order for the error e_h in the discrete L^2 norm is of an order higher than $\mathcal{O}(h)$.

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