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CLIPPING OVER DISSIPATION IN TURBULENCE MODELS

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We dedicate this work to Max Gunzburger.
He started us on this adventure and inspired us along the way.

Abstract. *Clipping* refers to adding 1 line of code $A \Leftarrow \min\{A, B\}$ to force the variable A to stay below a present bound B. Phenomenological clipping also occurs in turbulence models to correct for over dissipation caused by the action of eddy viscosity terms in regions of small scales. Herein we analyze eddy viscosity model energy dissipation rates with 2 phenomenological clipping strategies. Since the true Reynolds stresses are $O(d^2)$ (d = wall normal distance) in the near wall region, the first is to force this near wall behavior in the eddy viscosity by $\nu_{turb} \Leftarrow \min\{\nu_{turb}, \frac{\kappa}{T_{ref}} d^2\}$ for some preset κ and time scale T_{ref}. The second is Escudier's early proposal to clip the turbulence length scale in a common specification of ν_{turb}, reducing too large values in the interior of the flow. Analyzing respectively shear flow turbulence and turbulence in a box (i.e., periodic boundary conditions), we show that both clipping strategies do prevent aggregate over dissipation of model solutions.

Key words. Energy dissipation rate, turbulence.

1. Introduction

Clipping in scientific programming refers to adding 1 line of code to force a preset upper or lower bound such as $A \Leftarrow \min\{A, B\}$. As an example, the standard parameterization of an eddy viscosity coefficient is $\nu_{turb} = \mu l \sqrt{k}$ where μ is a constant, $l = l(x,t)$ is the model's turbulent length scale and $k = k(x,t)$ is the model's approximation to the turbulent kinetic energy. The $\sqrt{k}$ term in ν_{turb} is often implemented as $\sqrt{\max\{k, 0\}}$ clipping small negative k values. Phenomenologically deduced clipping occurs in turbulence models to correct for over dissipation caused by the action of eddy viscosity terms in regions of small velocity scales and is tested in numerical experiments. Herein we develop analytical support, analyzing model dissipation, for clipping in URANS (Unsteady Reynolds Averaged Navier Stokes) turbulence models, complementing phenomenology and numerical tests. The true Reynolds stresses are $\mathcal{O}(d^2)$ ($d = \inf\{|x - y| : y \in \partial\Omega\}$, the wall normal distance) in the near wall region. The first clipping strategy we analyze is to force this $\mathcal{O}(d^2)$ behavior in the eddy viscosity by $\nu_{turb} \Leftarrow \min\{\nu_{turb}, \frac{\kappa}{T_{ref}} d^2\}$ for some preset and non-dimensional κ and time scale T_{ref}. The second clipping strategy acts on the model's turbulence length scale, the variable l in $\nu_{turb} = \mu l \sqrt{k}$. We analyze Escudier's clipping of this turbulence length scale in the interior. Analyzing in the first and second cases respectively shear flow turbulence and turbulence in a box (i.e., periodic boundary conditions), we show that *both clipping strategies prevent aggregate over dissipation of eddy viscosity model solutions.*

A wide variety of eddy viscosity models exist. Current practice, summarized in Wilcox [40], favors eddy viscosity based, URANS models arising from time averaging, e.g., Durbin and Pettersson Reif [12] (p. 195). Following, for example Mohammadi and Pironneau [26] and Wilcox [40] (p.37 Eq 3.9), the model velocity

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$v(x,y,z,t) \simeq \overline{u}(x,y,z,t)$ approximates the finite time average[1] $\overline{u}$ of the Navier-Stokes velocity u

$$\overline{u}(x,y,z,t) = \frac{1}{\tau}\int_{t-\tau}^{t} u(x,y,z,t')dt' \text{ and fluctuation } u' := u - \overline{u}. \quad (1)$$

Causality requires the time window, $t-\tau < t' < t$, to stretch backwards as above so present velocities do not depend on future forces. The associated turbulent kinetic energy is then $\frac{1}{2}\overline{|u-\overline{u}|^2}$. Averaging the Navier Stokes equations (NSE) yields the system $\nabla \cdot \overline{u} = 0$ and

$$\overline{u}_t + \overline{u}\cdot\nabla\overline{u} - \nabla\cdot(2\nu\nabla^s\overline{u}) - \nabla\cdot R(u,u) + \nabla p = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,y,z,t')dt',$$
$$\text{where} \quad R(u,u) = \overline{u}\otimes\overline{u} - \overline{u\otimes u}.$$

Here ν is the kinematic viscosity, p is a pressure, f is the body force, $\nabla^s u$ is the symmetric part of ∇u, U is a global velocity scale, L is a global length scale and the Reynolds number is $\mathcal{R}e = LU/\nu$. This equation is not closed. Models replace $R(u,u)$ by terms that only depend on $\overline{u}$. For time window τ sufficiently large (and $t > \tau$) time dependence disappears from the equation and steady state RANS models result. For time window small, τ can be treated as a small parameter in $R(u,u)$ and models can be derived by asymptotics. Herein we consider URANS modelling for intermediate τ.

The main URANS model used in practical turbulent flow predictions is of eddy viscosity type. Its velocity $v(x,y,z,t) \simeq \overline{u}(x,y,z,t)$ satisfies

$$v_t + v\cdot\nabla v - \nabla\cdot(2[\nu+\nu_{turb}]\nabla^s v) + \nabla p = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,y,z,t')dt',\ \nabla\cdot v = 0. \quad (2)$$

Herein we first analyze in Section 2 the near wall behavior of the general eddy viscosity model, i.e., any choice of $\nu_{turb}(x,y,z,t) \geq 0$. The eddy or turbulent viscosity $\nu_{turb}(\geq 0)$ must be specified. Section 3 analyzes the away from wall behavior of the common, $1-$equation specification $\nu_{turb} = \mu l\sqrt{k}$ where $k(x,y,z,t)$ satisfies the classical equation for the turbulent kinetic energy.

A classical turbulent viscosity specification is the Smagorinsky-Ladyzhenskaya $0-$equation model $\nu_{turb} = (0.1\delta)^2|\nabla^s v|$ where δ = selected length scale, analyzed by Du, Gunzburger and Turner in [10], [36]. The classic $1-$equation model of Prandtl and Kolmogorov is analyzed in Section 3. $2-$equation models add a second, phenomenologically derived equation that determines the $1-$equation turbulence length scale l. In all these cases, the total *model energy dissipation rate per unit volume* is

$$\varepsilon_{\text{model}}(v) := \frac{1}{|\Omega|}\int_{\Omega} 2[\nu+\nu_{turb}]|\nabla^s v(x,y,z,t)|^2 dx. \quad (3)$$

A common failure mode of eddy viscosity models is over dissipation, either producing a lower $\mathcal{R}e$ flow or even driving the solution to a nonphysical steady state. This occurs due to the action of the turbulent viscosity term near walls or on interior small scales. We study over dissipation here through interrogation of the above model energy dissipation rate. A wide range of boundary conditions occur in practical flow simulations. Herein we focus on two: shear boundary conditions to study turbulence generated by near wall flows (Section 2) and $L-$periodic to study turbulence dynamics away from walls (Section 3).

[1]The time average can occur after ensemble averaging plus an ergodic hypothesis. URANS models are also constructed ad hoc simply by adding $\frac{\partial v}{\partial t}$ to a RANS model.

Section 2 studies clipping ν_{turb} near wall for general eddy viscosity models. The near wall behavior of the true Reynolds stress is $\nabla \cdot R(u,u) = \mathcal{O}(d^2)$. Matching this behavior in the model requires $\nu_{turb} = \mathcal{O}(d^2)$. Choosing the (dimensionless) constant κ and the reference time T_{ref}, this near wall asymptotics is enforceable through the clipping

$$\nu_{turb} \Leftarrow \min\{\nu_{turb}, \frac{\kappa}{T_{ref}} d^2\}.$$

The analysis of the effect of this near wall clipping on energy dissipation is performed in Section 2 for shear flows. Let ν_{eff} denote the effective viscosity (so $\frac{\nu}{\nu_{eff}} \leq 1$) and $\mathcal{R}e_{eff} = LU/\nu_{eff}$. We prove in Theorem 2.1 that this forced replication of the near wall asymptotics of the true Reynolds stresses does preclude model dissipation as long as $\kappa \leq \mathcal{O}(\mathcal{R}e_{eff})$. Theorem 3.1 asserts

$$\lim\sup_{T\to\infty} \frac{1}{T}\int_0^T \varepsilon_{\text{model}}(t)dt \leq \left[\frac{5}{2} + 32\frac{\nu}{\nu_{eff}} + \frac{\kappa}{6}\mathcal{R}e_{eff}^{-1}\left(\frac{T^*}{T_{ref}}\right)\right]\frac{U^3}{L}.$$

Section 3 studies Escudier's clipping of l away from walls when the eddy viscosity is determined through 1−equation models. The standard formulation of ν_{turb}, due to Prandtl and Kolmogorov, is $\nu_{turb} = \mu l\sqrt{k}$ where μ (typically 0.2 to 0.6) is a calibration constant, l is a turbulence length scale and $k \simeq \frac{1}{2}\overline{|u-\overline{u}|^2}$ is a model approximation to turbulent kinetic energy. Escudier observed that the traditional value $l = 0.41d$ is too large in the flow interior. Escudier [13], [14], see also [40] (p.78 Eq 3.108 and Ch. 3, p. 76 Eq 3.99), proposed clipping its maximum value (with the cap active away from walls) by

(4) $\quad l = \min\{0.41d, 0.09\delta\}$ where $\delta =$ estimate of transition region width.

In Section 3 we analyze the effect of this clipping in the interior of a turbulent flow via periodic boundary conditions. Theorem 3.1 establishes that over dissipation is again prevented

$$\lim\sup_{T\to\infty} \frac{1}{T}\int_0^T \varepsilon_{\text{model}}(t)dt \leq \left[3 + \frac{9}{2}\mathcal{R}e^{-1} + 0.03\mu^{3/2}\left(\frac{\delta}{L}\right)^2\right]\frac{U^3}{L}.$$

1.1. Previous work on model development. Saint-Venant [33] noted that turbulent mixing increases with "*the intensity of the whirling agitation*", [7] (p.235). Eddy viscosity models, based on the early work of Saint-Venant's student Boussinesq [1], are based on

Boussinesq:	*Turbulent fluctuations have a dissipative effect on the mean flow.*
EV hypothesis:	*This dissipation can be modelled by an eddy viscosity term* $\nabla \cdot (\nu_{turb}\nabla^s\overline{u})$.

Early work in the kinetic theory of gasses suggested the (dimensionally consistent) relation $\nu_{turb} = \frac{1}{3}lV$ where V is a velocity scale and l is an analog to a mean free pass. Prandtl and Kolmogorov noted that the enhanced mixing of turbulent flows is due to turbulent fluctuations and concluded that the correct velocity scale should be inferred from the turbulent kinetic energy $\frac{1}{2}\overline{|u-\overline{u}|^2}$. This reasoning led to the, now universally accepted (and dimensionally consistent), Kolmogorov-Prandtl relation $\nu_{turb} = \mu l\sqrt{k}$ where

$$l : \text{turbulence length scale and } k \ : \text{model approximation to } \frac{1}{2}\overline{|u-\overline{u}|^2}.$$

Pope [30] calculates $\mu = 0.55$ from the law of the wall. Davidson [6] (p. 114 Eq 4.11a) calculates $\mu \simeq 0.33$ in $2d$ and $\mu \simeq 0.27$ in $3d$ using a kinetic theory analogy. Prandtl and Kolmogorov, e.g., [31], [3] (p.99, Section 4.4) [6], [26] (p.60, Section 5.3) or [30] (p.369, Section 10.3) independently derived the following equation for the approximation to the turbulent kinetic energy

$$k_t + v \cdot \nabla k - \nabla \cdot ([\nu + \nu_{turb}] \nabla k) + \frac{1}{l} k\sqrt{k} = 2\nu_{turb}|\nabla^s v|^2. \tag{5}$$

The turbulence length scale l was postulated in 1915 by Taylor [34] and described by Prandtl [32] as the diameter of a coherent mass of fluid. The idea behind $l = 0.41d$ (among many variants [19]) was that near walls, this *diameter* of a coherent mass of fluid was constrained by the near wall distance. Away from walls, $l = 0.41d$ is too large and Escudier proposed the cap (4). In 1926 Prandtl [32] mentioned a second, kinematic possibility of the *distance a fluctuating eddy travels in one time unit*. This description motivated the choice $l = \sqrt{2}k^{1/2}\tau$ in [17], [35], [25]. Kolmogorov inferred l from a second equation, beginning the development of 2−equation models. There are many other proposed mixing lengths; the paper [19] studies 9 and describes more.

1.2. Previous work on energy dissipation rates. The energy dissipation rate is a fundamental statistic of turbulence, e.g., [30], [37]. The balance of Navier-Stokes equation's energy dissipation with energy input, $\varepsilon_{NSE} \simeq U^3/L$, is observed in physical experiments [15], [37]. In 1992 Constantin and Doering [5] established a direct link between phenomenology and NSE predicted energy dissipation. This work builds on [2], [16] (and others) and has developed in many important directions subsequently e.g., [8], [37], [38], [39].

Extending this work to turbulence models requires existence of weak solutions and a standard energy inequality. An existence theorem for weak solutions to a general eddy viscosity model is proven in [23] in which the uniform bound on ν_{turb} induced by clipping automatically enforces two of the three needed assumptions. The third depends on the specific dependence of ν_{turb} on v. The current state of existence theory is treated comprehensively in [3]. For some simple turbulence models, existence is known and á priori analysis has shown that $avg(\varepsilon_{\text{model}}) \leq \mathcal{O}(U^3/L)$, where the hidden constant does not blow up as $\mathcal{R}e \to \infty$, e.g., [11], [20], [21], [22], [24], [27], [28], [29]. For the 1-equation model with length scale $l = \sqrt{2}k^{1/2}\tau$ existence is plausible but still an open problem. Assuming existence and an energy inequality, this model has been proven in [25] not to over dissipate due to small scales generated by the nonlinearity. In the Smagorinsky model, Pakzad [27] has proven that wall damping functions, a clipping alternative, prevent over dissipation.

1.3. Notation and preliminaries. We assume that weak solutions of the systems studied exist and satisfy standard energy inequalities. In many cases this plausible assumption has not yet been proven, see [3] for current knowledge. The $L^2(\Omega)$ norm and the inner product are $\|\cdot\|$ and $(\cdot,\cdot)$. The $L^p(\Omega)$ norms are denoted $\|\cdot\|_{L^p}$. The contraction of two tensors is denoted as usual by a colon, $A : B = \sum_i \sum_j A_{ij}B_{ij}$. The absolute value of a scalar, Euclidean magnitude of a vector and tensor are denoted by $|\cdot|$. For example, $|v|^2 = \sum_i v_i^2$ and $|A|^2 = \sum_i \sum_j A_{ij}^2$. The volume element in integrals is denoted dx rather than $dxdydz$.

C represents a generic positive constant independent of $\nu, \mathcal{R}e$, other model parameters and the flow scales U, L defined below. In all cases the turbulent viscosity $\nu_{turb} = \nu_{turb}(x, y, z, t)$ and will be abbreviated by writing ν_{turb}.

Definition 1.1. The finite and long time averages of a function $\phi(t)$ are

$$\langle\phi\rangle_T = \frac{1}{T}\int_0^T \phi(t)dt \text{ and } \langle\phi\rangle = \lim\sup_{T\to\infty} \langle\phi\rangle_T .$$

These satisfy $\langle\langle\phi\rangle\rangle = \langle\phi\rangle$ and

$$\langle\phi\psi\rangle_T \leq \left\langle|\phi|^2\right\rangle_T^{1/2} \left\langle|\psi|^2\right\rangle_T^{1/2} \text{ and } \langle\phi\psi\rangle \leq \left\langle|\phi|^2\right\rangle^{1/2} \left\langle|\psi|^2\right\rangle^{1/2} . \tag{6}$$

2. Clipping ν_{turb} in the turbulent boundary layer

Over dissipation is often due to incorrect values of ν_{turb} in regions of small scales, i.e. where $\nabla^s v$ is large. These small scales are generated in the boundary layer and in the interior by breakdown of large scales through the nonlinearity. This section considers those generated predominantly in the turbulent boundary layer, studied via shear boundary conditions. Matching the near wall behavior $R(u,u) = \mathcal{O}(d^2)$ in the model's eddy viscosity term requires $\nu_{turb} = \mathcal{O}(d^2)$, enforced through the clipping

$$\nu_{turb} \Leftarrow \min\{\nu_{turb}, \frac{\kappa}{T_{ref}}d^2\} \tag{7}$$
$$\text{so that } 0 \leq \nu_{turb} \leq \frac{\kappa}{T_{ref}}d^2.$$

We study the effect of (7) via shear flows. Shear flows can develop several ways. Inflow boundary conditions can emulate a jet of water entering a vessel. A body force $f(\cdot)$ can be specified to be non-zero large and tangential at a fixed wall. The simplest (chosen herein) is a moving wall modelled by a boundary condition $v = g$ on the boundary where $g \cdot n = 0$. This setting includes flows between rotating cylinders. Select the flow domain $\Omega = (0, L)^3$, $L-$periodic boundary conditions in x, y, a fixed-wall no-slip condition at $z = 0$ and a wall at $z = L$ moving with velocity $(U, 0, 0)$:

$$\begin{array}{cc} \textit{Boundary} & \textit{Conditions}: \\ \text{moving top lid:} & v(x,y,L,t) = (U,0,0) \\ \text{fixed bottom wall:} & v(x,y,0,t) = (0,0,0) \\ \text{periodic side walls:} & \begin{array}{c} v(x+L,y,z,t) = v(x,y,z,t), \\ v(x,y+L,z,t) = v(x,y,z,t) \end{array} \end{array} \tag{8}$$

Herein, we assume that a weak solution of the model (2) with shear boundary conditions (8) exists and satisfies the usual energy inequality. Recall that the colon in $\nabla^s v : \nabla^s \phi$ below denotes the tensor contraction. Specifically, for any divergence free function $\phi(x, y, z)$ with $\phi, \nabla\phi \in L^2(\Omega)$ and satisfying the shear boundary conditions (8),

$$\frac{1}{2}\frac{d}{dt}||v||^2 + \int_\Omega 2[\nu + \nu_{turb}]|\nabla^s v|^2 dx \tag{9}$$
$$\leq (v_t, \phi) + \int_\Omega 2[\nu + \nu_{turb}]\nabla^s v : \nabla^s \phi dx + (v \cdot \nabla v, \phi).$$

To formulate our first main result we recall the definition of the **effective viscosity** ν_{eff} $(\geq \nu)$, well defined due to Proposition 2.3, and a few related quantities.

Definition 2.1. The *effective viscosity* ν_{eff} is

$$\nu_{eff} := \frac{\left\langle \frac{1}{|\Omega|}\int_\Omega [2\nu + 2\nu_{turb}]|\nabla^s v|^2 dx \right\rangle}{\left\langle \frac{1}{|\Omega|}\int_\Omega |\nabla^s v|^2 dx \right\rangle}.$$

The large scale turnover time is $T^* = L/U$. The *Reynolds number* and *effective Reynolds number* are $\mathcal{R}e = U\,L/\nu$ and $\mathcal{R}e_{eff} = U\,L/\nu_{eff}$. Let $\beta = \frac{1}{8}\mathcal{R}e_{eff}^{-1}$ and denote the region $\mathcal{S}_\beta$ by

$$\mathcal{S}_\beta = \{(x,y,z) : 0 \le x \le L, 0 \le y \le L, (1-\beta)L < z < L\}\,.$$

Theorem 2.2 asserts that ν_{eff} matching the near wall asymptotics of $R(u,u)$ is enough to ensure that the model does not over dissipate.

Theorem 2.1. *Assume* $0 \le \nu_{turb}(x,y,z,t) \le \frac{\kappa}{T_{ref}}d^2$*. Then, any weak solution of the eddy viscosity model (2) satisfying the energy inequality (9) has its model energy dissipation bounded as*

$$\langle \varepsilon_{model}\rangle \le \left[\frac{5}{2} + 32\frac{\nu}{\nu_{eff}} + \frac{\kappa}{6}\mathcal{R}e_{eff}^{-1}\left(\frac{T^*}{T_{ref}}\right)\right]\frac{U^3}{L}.$$

To begin the proof, we recall that uniform bounds follow from (9) by a known argument.

Proposition 2.1 (Uniform Bounds)**.** *Consider the model (2) with shear boundary conditions (8). Assume that there is a* $\kappa \ge 0$ *such that*

$$0 \le \nu_{turb}(x,y,z,t) \le \frac{\kappa}{T_{ref}}d^2.$$

Then, for a weak solution satisfying (9) the following are uniformly bounded in T

$$||v(T)||^2, \int_\Omega \nu_{turb}(\cdot,T)dx, \left\langle \int_\Omega |\nabla^s v|^2 dx\right\rangle_T, \left\langle \int_\Omega [2\nu + 2\nu_{turb}]|\nabla^s v|^2 dx\right\rangle_T.$$

Proof. Due to the clipping imposed we have $0 < \nu \le 2\nu + 2\nu_{turb} \le C < \infty$. Since $2\nu + 2\nu_{turb}$ is positive and uniformly bounded the above uniform bounds follow from differential inequalities exactly as in the NSE case and along the lines of the analogous proof in [18]. □

Proof of Theorem 2.2. Following Doering and Constantin [9], choose $\phi(z) = [\widetilde{\phi}(z), 0, 0]^T$ where

$$\widetilde{\phi}(z) = \begin{cases} 0, & z \in [0, L-\beta L] \\ \frac{U}{\beta L}(z-(L-\beta L)), & z \in [L-\beta L, L] \end{cases} \quad \beta = \frac{1}{8}\mathcal{R}e_{eff}^{-1}.$$

This function $\phi(z)$ is piecewise linear, continuous, divergence free and satisfies the boundary conditions. The following are easily calculated values

$$||\,\phi\,||_{L^\infty(\Omega)} = U, \qquad ||\,\nabla\phi\,||_{L^\infty(\Omega)} = \frac{U}{\beta L},$$
$$||\,\phi\,||^2 = \frac{1}{3}U^2\,\beta\,L^3, \qquad ||\,\nabla\,\phi\,||^2 = \frac{U^2 L}{\beta}.$$

With this choice of ϕ, time averaging the energy inequality (9) over $[0,T]$ and normalizing by $|\Omega| = L^3$ gives

$$\begin{aligned}(10)\qquad &\frac{1}{2TL^3}||v(T)||^2 + \left\langle \frac{1}{L^3}\int_\Omega [2\nu + 2\,\nu_{turb}]|\nabla^s v|^2 dx\right\rangle_T \\ \le &\frac{1}{2TL^3}||v(0)||^2 + \frac{1}{TL^3}(v(T)-v(0),\phi) + \left\langle \frac{1}{L^3}(v\cdot\nabla v,\phi)\right\rangle_T \\ &+ \left\langle \frac{1}{L^3}\int_\Omega [2\nu + 2\nu_{turb}]\nabla^s v : \nabla^s\phi dx\right\rangle_T.\end{aligned}$$

Due to the above á priori bounds the averaged energy inequality can be written as
(11)

$$\langle\varepsilon_{\text{model}}\rangle_T \leq \mathcal{O}(\frac{1}{T}) + \left\langle \frac{1}{L^3}(v\cdot\nabla v,\phi)\right\rangle_T + \left\langle \frac{1}{L^3}\int_\Omega [2\nu + 2\,\nu_{turb}]\nabla^s v : \nabla^s\phi dx\right\rangle_T.$$

The main issue is thus the third term, $\int 2\nu_{turb}\nabla^s v : \nabla^s \phi dx$. Before treating that we recall the analysis of Doering and Constantine [9] and Wang [38] for the two terms shared by the NSE, $(v\cdot\nabla v,\phi)$ and $\int 2\nu\nabla^s v : \nabla^s\phi dx$. For the nonlinear term $\left\langle \frac{1}{L^3}(v\cdot\nabla v,\phi)\right\rangle_T =: NLT$, we have

$$\begin{aligned}
NLT =& \left\langle \frac{1}{L^3}(v\cdot\nabla v,\phi)\right\rangle_T = \left\langle \frac{1}{L^3}([v-\phi]\cdot\nabla v,\phi)\right\rangle_T + \left\langle \frac{1}{L^3}(\phi\cdot\nabla v,\phi)\right\rangle_T \\
\leq& \left\langle \frac{1}{L^3}\int_{\mathcal{S}_\beta} |v-\phi||\nabla v||\phi| + |\phi|^2|\nabla v| dx\right\rangle_T \\
\leq& \frac{1}{L^3}\left\langle \left\|\frac{v-\phi}{L-z}\right\|_{L^2(\mathcal{S}_\beta)} ||\nabla v||_{L^2(\mathcal{S}_\beta)}||(L-z)\phi||_{L^\infty(\mathcal{S}_\beta)} \right. \\
&\left. + ||\phi||^2_{L^\infty(\mathcal{S}_\beta)}||\nabla v||_{L^1(\mathcal{S}_\beta)}\right\rangle_T.
\end{aligned}$$

On the RHS, $||\phi||^2_{L^\infty(\mathcal{S}_\beta)} = U^2$and $||(L-z)\phi||_{L^\infty(\mathcal{S}_\beta)} = \frac{1}{4}\beta LU$. Since $v-\phi$ vanishes on $\partial\mathcal{S}_\beta$, Hardy's inequality, the triangle inequality and a calculation imply

$$\begin{aligned}
\left\|\frac{v-\phi}{L-z}\right\|_{L^2(\mathcal{S}_\beta)} &\leq 2\left\|\nabla(v-\phi)\right\|_{L^2(\mathcal{S}_\beta)} \leq 2\left\|\nabla v\right\|_{L^2(\mathcal{S}_\beta)} + 2\left\|\nabla\phi\right\|_{L^2(\mathcal{S}_\beta)} \\
&\leq 2\left\|\nabla v\right\|_{L^2(\mathcal{S}_\beta)} + 2U\sqrt{\frac{L}{\beta}}.
\end{aligned}$$

Thus we have the estimate
(12)

$$NLT \leq \frac{\beta LU}{4}\frac{1}{L^3}\left\langle 2||\nabla v||^2_{L^2(\mathcal{S}_\beta)} + 2U\sqrt{\frac{L}{\beta}}||v||_{L^2(\mathcal{S}_\beta)}\right\rangle_T + \frac{U^2}{L^3}\left\langle ||\nabla v||_{L^1(\mathcal{S}_\beta)}\right\rangle_T.$$

For the last term on the RHS, Hölders inequality in space then in time implies

$$\begin{aligned}
\frac{U^2}{L^3}\left\langle ||\nabla v||_{L^1(\mathcal{S}_\beta)}\right\rangle_T &= \frac{U^2}{L^3}\left\langle \int_{\mathcal{S}_\beta} |\nabla v|\cdot 1 dx\right\rangle_T \\
&\leq \frac{U^2}{L^3}\left\langle \sqrt{\int_{\mathcal{S}_\beta}|\nabla v|^2 dx}\sqrt{\beta L^3}\right\rangle_T \\
&\leq \frac{U^2\sqrt{\beta}}{L^{3/2}}\left\langle \sqrt{\int_{\mathcal{S}_\beta}|\nabla v|^2 dx}\right\rangle_T \\
&\leq \frac{U^2\sqrt{\beta}}{L^{3/2}}\left\langle \int_{\mathcal{S}_\beta}|\nabla v|^2 dx\right\rangle_T^{1/2}.
\end{aligned}$$

Increase the integral from $\mathcal{S}_\beta$ to Ω, use (as $\nabla \cdot v = 0$) $||\nabla v||^2 = 2||\nabla^s v||^2$ and $\beta = \frac{1}{8}\mathcal{R}e_{eff}^{-1}$. Rearranging and using the arithmetic-geometric inequality gives

$$\begin{aligned}\frac{U^2}{L^3}\left\langle ||\nabla v||_{L^1(\mathcal{S}_\beta)}\right\rangle_T \leq & U^2\sqrt{\beta}\left\langle \frac{1}{L^3}\int_\Omega 2|\nabla^s v|^2 dx\right\rangle_T^{1/2} \\ \leq & U^2\sqrt{\frac{2}{8}\frac{1}{LU}}\left\langle \frac{1}{L^3}\int_\Omega \nu_{eff}|\nabla^s v|^2 dx\right\rangle_T^{1/2} \\ \leq & \left(\frac{U^3}{L}\right)^{1/2}\frac{1}{2}\left\langle \frac{1}{L^3}\int_\Omega \nu_{eff}|\nabla^s v|^2 dx\right\rangle_T^{1/2} \\ \leq & \frac{1}{2}\frac{U^3}{L} + \frac{1}{8}\left\langle \frac{1}{L^3}\int_\Omega \nu_{eff}|\nabla^s v|^2 dx\right\rangle_T .\end{aligned}$$

Similar manipulations yield

$$\begin{aligned}\frac{1}{4}\beta LU\frac{1}{L^3}\left\langle 2U\sqrt{\frac{L}{\beta}}||v||_{L^2(\mathcal{S}_\beta)}\right\rangle_T \leq & \frac{1}{2}\beta LU\left\langle \frac{1}{L^3}||\nabla v||^2_{L^2(\mathcal{S}_\beta)}\right\rangle_T + \frac{1}{8}\frac{U^3}{L} \\ \leq & \frac{1}{8}\left\langle \frac{1}{L^3}\nu_{eff}||\nabla^s v||^2_{L^2(\mathcal{S}_\beta)}\right\rangle_T + \frac{1}{8}\frac{U^3}{L}.\end{aligned}$$

Using the last two estimates in the NLT upper bound (12), we obtain

$$NLT \leq 2\beta\frac{LU}{\nu_{eff}}\left\langle \frac{1}{L^3}\nu_{eff}||\nabla^s v||^2_{L^2(\mathcal{S}_\beta)}\right\rangle_T + \frac{5}{8}\frac{U^3}{L}.$$

Thus,

$$\begin{aligned}\langle\varepsilon_{\text{model}}\rangle_T \leq & \mathcal{O}(\frac{1}{T}) + \frac{1}{4}\left\langle \frac{1}{L^3}\nu_{eff}||\nabla^s v||^2_{L^2(\Omega)}\right\rangle_T + \frac{5}{8}\frac{U^3}{L} \\ & + \left\langle \frac{1}{L^3}\int_\Omega 2[\nu + \nu_{turb}]\nabla^s v : \nabla^s\phi dx\right\rangle_T .\end{aligned}$$

Consider now the last term on the RHS. Since ϕ is zero off $\mathcal{S}_\beta$,

$$\begin{aligned}&\left\langle \frac{1}{L^3}\int_\Omega 2[\nu + \nu_{turb}]\nabla^s v : \nabla^s\phi dx\right\rangle_T \\ =& \left\langle \frac{1}{L^3}\int_{\mathcal{S}_\beta} 2[\nu + \nu_{turb}]\nabla^s v : \nabla^s\phi dx\right\rangle_T \\ \leq & \frac{1}{2}\langle\varepsilon_{\text{model}}\rangle_T + \frac{1}{2}\left\langle \frac{1}{L^3}\int_{\mathcal{S}_\beta} 2[\nu + \nu_{turb}]\left(\frac{U}{\beta L}\right)^2 dx\right\rangle_T \\ \leq & \frac{1}{2}\langle\varepsilon_{\text{model}}\rangle_T + \frac{1}{2}\left(\frac{U}{\beta L}\right)^2\beta\left\langle \frac{1}{\beta L^3}\int_{\mathcal{S}_\beta} 2[\nu + \nu_{turb}]dx\right\rangle_T .\end{aligned}$$

Thus, as $\beta = \frac{1}{8}\mathcal{R}e_{eff}^{-1}$ implies $2\beta\mathcal{R}e_{eff} = 1/4$,

$$\begin{aligned}\frac{1}{2}\langle\varepsilon_{\text{model}}\rangle_T \leq & \mathcal{O}(\frac{1}{T}) + \frac{1}{4}\left\langle \frac{1}{L^3}\nu_{eff}||\nabla^s v||^2_{L^2(\Omega)}\right\rangle_T \\ & + \frac{5}{8}\frac{U^3}{L} + \frac{\beta}{2}\left(\frac{U}{\beta L}\right)^2\left\langle \frac{1}{\beta L^3}\int_{\mathcal{S}_\beta} 2\nu + 2\nu_{turb}dx\right\rangle_T .\end{aligned}$$

As a subsequence $T_j \to \infty$

$$\left\langle \frac{1}{L^3}\nu_{eff}||\nabla^s v||^2_{L^2(\Omega)} \right\rangle_T \to \langle \varepsilon_{\text{model}} \rangle .$$

For the other term we calculate

$$\left\langle \frac{1}{\beta L^3}\int_{\mathcal{S}_\beta} 2\nu_{turb} dx \right\rangle_T \leq \left\langle \frac{1}{\beta L^3}\int_{\mathcal{S}_\beta} \frac{2\kappa}{T_{ref}} d^2 dx \right\rangle_T = \frac{2\kappa}{3T_{ref}}\beta^2 L^2$$
$$= \frac{2\kappa}{3T_{ref}} L^2 \left(\frac{1}{8}\mathcal{R}e_{eff}^{-1}\right)^2 = \frac{2\kappa L^2}{192 T_{ref}}\mathcal{R}e_{eff}^{-2}$$

Thus,

$$\frac{1}{2}\langle \varepsilon_{\text{model}} \rangle \leq \frac{1}{4}\langle \varepsilon_{\text{model}} \rangle + \frac{5}{8}\frac{U^3}{L} + \frac{1}{2}\frac{U^2}{\beta L^2}\left[2\nu + \frac{2\kappa L^2}{192 T_{ref}}\mathcal{R}e_{eff}^{-2}\right].$$

Next, insert $\beta = \frac{1}{8}\mathcal{R}e_{eff}^{-1} = \frac{1}{8}\frac{\nu_{eff}}{LU}$ and rewrite $1/T_{ref} = (T^*/T_{ref}) \cdot (1/T^*) = (T^*/T_{ref}) \cdot (U/L)$. This, after simplification, completes the proof

$$\langle \varepsilon_{\text{model}} \rangle \leq \left[\frac{5}{2} + 32\frac{\nu}{\nu_{eff}} + \frac{\kappa}{6}\mathcal{R}e_{eff}^{-1}\left(\frac{T^*}{T_{ref}}\right)\right]\frac{U^3}{L}.$$

□

3. Escudier's clipping of l away from walls

We analyze in this section the effect, away from walls, of the clipping developed by Escudier [13], [14], and still current practice [40] (p.78, Eq3.108 and Ch. 3, p. 76, Eq 3.99)

$$l \Leftarrow \min\{l, 0.09\delta\} \text{ where } \delta = \text{estimate of transition layer width}$$

which implies $0 \leq l \leq 0.09\delta$. We prove below that for problems without boundary layers (studied through periodic boundary conditions) this cap ensures that energy dissipation rates scale correctly. Thus such caps are an effective tool for precluding aggregate over dissipation due to the action of eddy viscosity in regions of interior small scales. Escudier's proposal (and our analysis) is for the 1−equation model of Prandtl and Kolmogorov, see also [3] (p.99, Section 4.4) [6], [26] (p.60, Section 5.3) or [30] (p.369, Section 10.3), given by

$$\begin{gathered} \nabla \cdot v = 0, \\ v_t + v\cdot\nabla v - \nabla\cdot([2\nu + 2\nu_{turb}]\nabla^s v) + \nabla p = f(x,y,z), \\ k_t + v\cdot\nabla k - \nabla\cdot([\nu+\nu_{turb}]\nabla k) + \frac{1}{l}k\sqrt{k} = 2\nu_{turb}|\nabla^s v|^2 \\ \text{where } \nu_{turb} = \mu l\sqrt{k}, \text{ with } \mu \simeq 0.55, \text{ and } 0 \leq l \leq 0.09\delta. \end{gathered} \tag{13}$$

The flow domain is $\Omega = (0, L_\Omega)^3$. Since the effect of the clipped value is in the flow interior, we impose $L_\Omega-$periodic boundary conditions on $\phi = k(x,y,z,t)$ and $L_\Omega-$periodic with zero mean boundary conditions on $\phi = v, p, v_0, f$:

$$\textit{Periodic}: \begin{cases} \phi(x+L_\Omega,y,z,t) = \phi(x,y,z,t) \\ \phi(x,y+L_\Omega,z,t) = \phi(x,y,z,t) \\ \phi(x,y,z+L_\Omega,t) = \phi(x,y,z,t) \end{cases} \text{ and } \textit{Zero mean}: \int_\Omega \phi dx = 0 . \tag{14}$$

The global length scale L must reflects the scales where the body force is inputting energy. Define the global velocity scale U, the body force scale F and large length scale L by

$$\left.\begin{array}{c} F = \left(\frac{1}{|\Omega|}\int_\Omega |f(x,y,z)|^2 dx\right)^{1/2}, \\ L = \min\left[L_\Omega, \frac{F}{\sup_{(x,y,z)\in\Omega}|\nabla^s f(x,y,z)|}, \frac{F}{\left(\frac{1}{|\Omega|}\int_\Omega |\nabla^s f(x,y,z)|^2 dx\right)^{1/2}}\right] \\ U = \left\langle \frac{1}{|\Omega|}\int_\Omega |v(x,y,z,t)|^2 dx\right\rangle^{1/2}. \end{array}\right\} \tag{15}$$

L has units of length and satisfies

$$||\nabla^s f||_\infty \le \frac{F}{L} \text{ and } \frac{1}{|\Omega|}||\nabla^s f||^2 \le \frac{F^2}{L^2}\,. \tag{16}$$

The standard energy inequality and equality for this system are

$$\left.\begin{array}{c} \frac{d}{dt}\frac{1}{|\Omega|}\frac{1}{2}||v||^2 + \frac{1}{|\Omega|}\int_\Omega [2\nu + 2\nu_{turb}]|\nabla^s v(x,y,z,t)|^2 dx \le \frac{1}{|\Omega|}(f,v), \\ \frac{d}{dt}\int_\Omega k dx + \int_\Omega \frac{1}{l}k\sqrt{k}dx = \int_\Omega 2\nu_{turb}|\nabla^s v|^2 dx. \end{array}\right\} \tag{17}$$

Since $l \le 0.09\delta$ the following two inequalities hold

$$\varepsilon_{\text{model}}(t) = \frac{1}{|\Omega|}\int_\Omega 2[\nu + \nu_{turb}]|\nabla^s v|^2 dx \le \frac{1}{|\Omega|}\int_\Omega 2\left[\nu + \mu 0.09\delta\sqrt{k}\right]|\nabla^s v|^2 dx,$$

$$(0.09\delta)^{-1}\left\langle \int_\Omega k^{3/2}dx\right\rangle \le \left\langle \int_\Omega \frac{1}{l}k\sqrt{k}dx\right\rangle = \left\langle \int_\Omega 2\nu_{turb}|\nabla^s v|^2 dx\right\rangle. \tag{18}$$

Theorem 3.1. *Consider the $1-$equation model under periodic with zero mean boundary conditions with $0 \le l \le 0.09\delta$. The time averaged energy dissipation rate of any weak solution satisfying the energy inequality (17) is bounded by*

$$\langle \varepsilon_{model}\rangle \le \left[3 + \frac{9}{2}\mathcal{R}e^{-1} + 0.03\mu^{3/2}\left(\frac{\delta}{L}\right)^2\right]\frac{U^3}{L}.$$

Proof. The following uniform in T bounds follow from the energy inequalities and $l \le 0.09\delta$ by differential inequalities as in [18]

$$\begin{array}{c} \frac{1}{2}||v(T)||^2 + \int_\Omega k(T)dx \le C < \infty\ , \\ \frac{1}{T}\int_0^T\int_\Omega \nu|\nabla^s v|^2 + \nu_{turb}|\nabla^s v|^2 + k^{3/2}dxdt \le C < \infty. \end{array} \tag{19}$$

Time averaging the energy inequality (17) and using the above á priori bounds and the Cauchy-Schwarz inequality gives

$$\mathcal{O}\left(\frac{1}{T}\right) + \langle\varepsilon_{\text{model}}\rangle_T \le F\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}}. \tag{20}$$

(Since we will let $T \to \infty$ the size of the hidden constant in $\mathcal{O}(1/T)$ does not matter. It would for finite time averages.) To bound F in terms of flow quantities, take the $L^2(\Omega)$ inner product of the model momentum equation with f, integrate by parts and average over $[0,T]$. This gives

$$\begin{aligned} F^2 =& \frac{1}{T}\frac{1}{|\Omega|}(v(T) - v_0, f) - \left\langle \frac{1}{|\Omega|}(vv, \nabla^s f)\right\rangle_T \\ &+ \left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu\nabla^s v : \nabla^s f + 2\nu_{turb}\nabla^s v : \nabla^s f dx\right\rangle_T. \end{aligned} \tag{21}$$

The term $\frac{1}{T}\frac{1}{|\Omega|}(v(T)-v_0,f)$ on the RHS is $\mathcal{O}(1/T)$. The second term is bounded by the Cauchy-Schwarz inequality and (16) by

$$\begin{aligned}\left|\left\langle \frac{1}{|\Omega|}(vv,\nabla^s f)\right\rangle_T\right| &\leq \left\langle ||\nabla^s f||_\infty \frac{1}{|\Omega|}||vv||^2\right\rangle_T \\ \leq ||\nabla^s f||_\infty \left\langle \frac{1}{|\Omega|}||v(\cdot,t)||^2\right\rangle_T &\leq \frac{F}{L}\left\langle \frac{1}{|\Omega|}||v(\cdot,t)||^2\right\rangle_T .\end{aligned}$$

The third term is bounded analogously

$$\begin{aligned}\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu\nabla^s v(x,y,z,t):\nabla^s f(x,y,z)dx\right\rangle_T &\leq \left\langle \frac{4\nu^2}{|\Omega|}||\nabla^s v||^2\right\rangle_T^{\frac{1}{2}} \left\langle \frac{1}{|\Omega|}||\nabla^s f||^2\right\rangle_T^{\frac{1}{2}} \\ &\leq \left\langle \frac{2\nu}{|\Omega|}||\nabla^s v||^2\right\rangle^{\frac{1}{2}} \frac{\sqrt{2\nu}F}{L} \\ &\leq \frac{\beta F}{2U}\left\langle \frac{2\nu}{|\Omega|}||\nabla^s v||^2\right\rangle_T + \frac{1}{\beta}\frac{\nu UF}{L^2},\end{aligned}$$

for any $0<\beta<1$. The fourth term's estimation is by successive applications of the space then time Cauchy-Schwarz inequality as follows

$$\begin{aligned}&\left|\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}\nabla^s v(x,y,z,t):\nabla^s f(x)dx\right\rangle_T\right| \\ &\leq \left\langle \frac{1}{|\Omega|}\int_\Omega \left(\sqrt{2\nu_{turb}}\right)\left(\sqrt{2\nu_{turb}}|\nabla^s v|\right)|\nabla^s f|dx\right\rangle_T \\ &\leq ||\nabla^s f||_\infty \left\langle \left(\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right)^{\frac{1}{2}} \left(\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}|\nabla^s v|^2 dx\right)^{\frac{1}{2}} dx\right\rangle_T \\ &\leq \frac{F}{L}\left\langle \frac{U}{F}\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle_T^{\frac{1}{2}} \left\langle \frac{F}{U}\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}|\nabla^s v|^2 dx\right\rangle_T^{\frac{1}{2}}.\end{aligned}$$

The arithmetic-geometric mean inequality then implies

$$\begin{aligned}&\left|\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}\nabla^s v(x,y,z,t):\nabla^s f(x,y,z)dx\right\rangle_T\right| \\ &\leq \frac{\beta}{2}\frac{F}{U}\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}|\nabla^s v|^2 dx\right\rangle_T + \frac{1}{2\beta}\frac{UF}{L^2}\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle_T .\end{aligned}$$

Using these four estimates in the bound for F^2 yields

$$\begin{aligned}F^2 \leq &\mathcal{O}\left(\frac{1}{T}\right) + \frac{F}{L}\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T + \frac{1}{2\beta}\frac{UF}{L^2}\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle_T \\ &+ \frac{1}{\beta}\frac{\nu UF}{L^2} + \frac{\beta F}{2U}\langle \varepsilon_{\text{model}}\rangle_T .\end{aligned}$$

Thus, we have the estimate

$$\begin{aligned}F\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}} \leq &\mathcal{O}\left(\frac{1}{T}\right) + \frac{1}{L}\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{3}{2}} \\ &+ \frac{\beta}{2}\frac{\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}}}{U}\langle \varepsilon_{\text{model}}\rangle_T + \frac{1}{2\beta}\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}} \frac{2\nu U}{L^2} \\ &+ \frac{1}{2\beta}\left\langle \frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}} \frac{U}{L^2}\left\langle \frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle_T .\end{aligned}$$

Inserting this on the RHS of (20) yields

$$\begin{aligned}\langle\varepsilon_{\text{model}}\rangle_T \leq &\mathcal{O}\left(\frac{1}{T}\right)+\frac{1}{L}\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{3}{2}} \\ &+\frac{\beta}{2}\frac{\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}}}{U}\langle\varepsilon_{\text{model}}\rangle_T+\frac{1}{2\beta}\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}}U\frac{2\nu}{L^2} \\ &+\frac{1}{2\beta}\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{\frac{1}{2}}\frac{U}{L^2}\left\langle\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle_T.\end{aligned} \tag{22}$$

The last term on the RHS is bounded using Hölder's inequality as

$$\begin{aligned}\left\langle\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle_T =&\left\langle\frac{1}{|\Omega|}\int_\Omega 2\mu l\sqrt{k}dx\right\rangle_T \leq 2\mu 0.09\delta\left\langle\frac{1}{|\Omega|}\int_\Omega 1\cdot\sqrt{k}dx\right\rangle_T \\ \leq& 2\mu 0.09\delta\frac{1}{T}\int_0^T\left(\frac{1}{|\Omega|}\int_\Omega k^{3/2}dx\right)^{1/3}dt \\ \leq& 2\mu 0.09\delta\left\langle\frac{1}{|\Omega|}\int_\Omega k^{3/2}dx\right\rangle_T^{1/3}.\end{aligned}$$

The second equation of (17), the integrated $k-$equation, states

$$\frac{d}{dt}\int_\Omega kdx+\int_\Omega\frac{1}{l}k^{3/2}dx=\int_\Omega 2\nu_{turb}|\nabla^s v|^2dx. \tag{23}$$

Time averaging the above gives

$$\mathcal{O}\left(\frac{1}{T}\right)+\frac{1}{T}\int_0^T\frac{1}{|\Omega|}\int_\Omega\frac{1}{l}k^{3/2}dxdt=\frac{1}{T}\int_0^T\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}|\nabla^s v|^2dxdt.$$

Thus,

$$(0.09\delta)^{-1}\left\langle\int_\Omega k^{3/2}dx\right\rangle\leq\left\langle\int_\Omega\frac{1}{l}k\sqrt{k}dx\right\rangle=\left\langle\int_\Omega 2\nu_{turb}|\nabla^s v|^2dx\right\rangle$$

and therefore, using (18),

$$\begin{aligned}\left\langle\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}dx\right\rangle&\leq\mathcal{O}\left(\frac{1}{T}\right)+2\mu 0.09\delta\left\langle\frac{1}{|\Omega|}\int_\Omega k^{3/2}dx\right\rangle^{1/3} \\ &\leq\mathcal{O}\left(\frac{1}{T}\right)+2\mu(0.09\delta)^{4/3}\left\langle\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}|\nabla^s.v|^2dx\right\rangle^{1/3}\end{aligned}$$

Assembling the above pieces we have

$$\begin{aligned}\langle\varepsilon_{\text{model}}\rangle_T\leq&\mathcal{O}\left(\frac{1}{T}\right)+\frac{1}{L}\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{3/2}+\frac{\beta}{2}\frac{\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{1/2}}{U}\langle\varepsilon_{\text{model}}\rangle_T \\ &+\frac{1}{2\beta}\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{1/2}U\frac{2\nu}{L^2} \\ &+\frac{1}{2\beta}\left\langle\frac{1}{|\Omega|}||v||^2\right\rangle_T^{1/2}\frac{U}{L^2}2\mu(0.09\delta)^{4/3}\left\langle\frac{1}{|\Omega|}\int_\Omega 2\nu_{turb}|\nabla^s v|^2dx\right\rangle^{1/3}.\end{aligned}$$

Let $T_j \to \infty$,

$$\begin{aligned}\langle \varepsilon_{\text{model}} \rangle \leq & \frac{U^3}{L} + \frac{\beta}{2} \langle \varepsilon_{\text{model}} \rangle + \frac{1}{2\beta} U^2 \frac{2\nu}{L^2} \\ & + \frac{1}{2\beta} \frac{U^2}{L^2} 2\mu (0.09\delta)^{4/3} \left\langle \frac{1}{|\Omega|} \int_\Omega 2\nu_{turb} |\nabla^s v|^2 dx \right\rangle^{1/3},\end{aligned}$$

which implies

$$\langle \varepsilon_{\text{model}} \rangle \leq \frac{U^3}{L} + \frac{\beta}{2} \langle \varepsilon_{\text{model}} \rangle + \frac{1}{2\beta} U^2 \frac{2\nu}{L^2} + \left(\frac{1}{2\beta} \frac{U^2}{L^2} 2\mu (0.09\delta)^{4/3} \right) \langle \varepsilon_{\text{model}} \rangle^{1/3}.$$

For the last term on the RHS use $ab \leq \frac{2}{3} a^{3/2} + \frac{1}{3} b^3$

$$\begin{aligned}\langle \varepsilon_{\text{model}} \rangle \leq & \frac{U^3}{L} + \frac{\beta}{2} \langle \varepsilon_{\text{model}} \rangle + \frac{1}{2\beta} U^2 \frac{2\nu}{L^2} \\ & + \frac{2}{3} \left(\frac{1}{2\beta} \frac{U^2}{L^2} 2\mu (0.09\delta)^{4/3} \right)^{3/2} + \frac{1}{3} \langle \varepsilon_{\text{model}} \rangle .\end{aligned}$$

Note that $U^2 \frac{\nu}{L^2} = \frac{U^3}{L} \frac{\nu}{LU} = \mathcal{R}e^{-1} \frac{U^3}{L}$. We then have, picking $\beta = 2/3$, collecting like terms and simplifying,

$$\langle \varepsilon_{\text{model}} \rangle \leq \left[3 + \frac{9}{2} \mathcal{R}e^{-1} + 0.03 \mu^{3/2} \left(\frac{\delta}{L} \right)^2 \right] \frac{U^3}{L}.$$

□

4. Conclusions and open problems

The upper bounds for energy dissipation rates of Doering, Constantin and Foias for the NSE were a breakthrough, connecting turbulence phenomenology with rigorous mathematical analysis. The recent result of Chow and Pakzad [4] that expected values of dissipation rates are in some regimes bounded *below* uniformly in the Reynolds number is a significant extension of this analysis. For turbulence models an upper bound of $\mathcal{O}(U^3/L)$, where the hidden constant does not blow up as $\mathcal{R}e \to \infty$, directly addresses a question of computational practice since this estimate precludes a common failure for the specific model analyzed. Since these are upper bounds, a bound where the hidden constant blows up as $\mathcal{R}e \to \infty$, while suggestive of over dissipation, must be complemented by numerical tests checking if the upper estimate is sharp.

For $1-$equation or $2-$equation eddy viscosity models and more general flow problems than shear flows and turbulence in a box, our results suggest combining an interior cap on the turbulence length scale and a near wall cap on ν_{turb} will be effective in many cases. This means that eddy viscosities ν_{turb} developed for model accuracy (possibly involving many calibration constants or using machine learning tools) can be forced to have a correct energy dissipation balance by essentially 2 extra lines of code.

The analysis herein does not include interior shear flows, as when a jet of fluid enters a large tank, and likely other cases. These are important open problems. Another common modelling technique is to relax the no slip condition at walls by a slip with friction / Navier slip law where the friction is a model calibration coefficient. Analysis of these cases is also an important open problem.

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