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POLYNOMIAL PRESERVING GRADIENT RECOVERY AND *A POSTERIORI* ESTIMATE FOR BILINEAR ELEMENT ON IRREGULAR QUADRILATERALS

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Abstract. A polynomial preserving gradient recovery method is proposed and analyzed for bilinear element under quadrilateral meshes. It has been proven that the recovered gradient converges at a rate $O(h^{1+\rho})$ for $\rho = \min(\alpha, 1)$, when the mesh is distorted $O(h^{1+\alpha})$ $(\alpha > 0)$ from a regular one. Consequently, the *a posteriori* error estimator based on the recovered gradient is asymptotically exact.

Key Words. Finite element method, quadrilateral mesh, gradient recovery, superconvergence, *a posteriori* error estimate.

1. Introduction

A posteriori error estimation is an active research area and many methods have been developed. Roughly speaking, there are residual type error estimators and recovery type estimators. For the literature, readers are referred to recent books by Ainsworth-Oden [2] and by Babuška-Strouboulis [4], a conference proceeding [16], a survey article by Bank [5], an earlier book by Verfürth [23], and references therein.

While residual type estimators have been analyzed extensively, there is only limited theoretical research on recovery type error estimators (see, e.g., [2, Chapter 4], [6, 7, 9, 10, 15, 22, 28, 29]). Yet, recovery type error estimators are widely used in engineering applications and their practical effectiveness has been recognized by more and more researchers. Currently, ZZ patch recovery is used in commercial codes, such as ANSYS, MCS/NASTRAN-Marc, Pro/MECHANICA (a product of Parametric Technology), and I-DEAS (a product of SDRC, part of EDS), for the purpose of smoothing and adaptive re-meshing. It is also used in NASA's COMET-AR (COmputational MEchanics Testbed With Adaptive Refinement). In a computer based investigation [4] by Babuška et al., it was found that among all error estimators tested (including the equilibrated residual error estimator, the ZZ patch recovery error estimator, and many others), the ZZ patch recovery error estimator based on the discrete least-squares fitting is the most robust.

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It is worth pointing out that the recovery type error estimator was originally based on finite element superconvergence theory, in hopes that a recovered gradient was superconvergent and hence could be used as a substitute of the exact gradient to measure the error. The reader is referred to [4, 11, 16, 18, 25, 33] for literature regarding superconvergence theory. In order to prove superconvergence, it is necessary to impose some strong restrictions on mesh, which are usually not satisfied in practice. Nevertheless, it is found that in many practical situations, recovery type error estimators perform astonishingly well under meshes produced by the Delaunay triangulation. Mathematically, this fact has not yet been rigorously justified.

In a recent work, Bank-Xu [6, 7] introduced a recovery type error estimator based on global L_2-projection with smoothing iteration of the multigrid method, and they established asymptotic exactness in the H^1-norm for linear element under shape regular triangulation. However, the recovery operator is a global one.

On the other hand, Wang proposed a "semi-local" recovery [27] and proved its superconvergence under the quasi-uniform mesh assumption. The main feature of his method is to apply L_2 projection on a coarser mesh with size $\tau = Ch^\alpha$ with $\alpha \in (0, 1)$. Consequently, there is no upper bound for the number of elements in an element patch when mesh size $h \to 0$.

As for element-wise recovery operators, Schatz-Wahlbin et al. [15, 22] established a general framework which requests, for linear element, given a fixed $0 < \epsilon < 1$, that

$$m = C\left(\left(\frac{H}{h}\right)^2 h^\epsilon + \left(\frac{h}{H}\right)^\epsilon \ln\frac{H}{h}\right) < 1.$$

Here h is the size of element τ, $H \geq 2h$ is the size of the patch ω_τ (surrounding τ), where the recovery takes place, and C is an unknown constant which comes from the analysis. Let $H = Lh$. In order for $m < 1$, we need

$$C(L^2 h^\epsilon + L^{-\epsilon}\ln L) < 1.$$

Depending on C, this essentially asks for sufficiently large L and sufficiently small h, which implies many elements may be needed for the recovery operator. Nevertheless, in practice, many recovery operators work well with an H/h that is not large (usually 2).

Therefore a theoretical justification for recovery that involves only a few elements surrounding a node is necessary. In other word, it is desired to study the case when $H = 2h$. The situation is further complicated by quadrilateral meshes where mappings between the reference element and physical elements are not affine. We encounter some delicate theoretical issue in analysis. See [1, 3, 8, 13, 14, 19, 21, 30, 31, 36] for more details.

In this article, we propose and analyze a gradient recovery method which is different from the ZZ recovery [34]. We show that the *a posteriori* estimate based on this new recovery operator is asymptotically exact under mesh distortion $O(h^{1+\alpha})$ when $\alpha > 0$. Here $\alpha = \infty$ represents the uniform mesh and $\alpha = 0$ represents completely unstructured mesh.

The main feature of this new recovery operator is: