INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 1, Number 1, Pages 49–64

LEAST-SQUARES FINITE ELEMENT METHODS FOR FIRST-ORDER ELLIPTIC SYSTEMS

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Abstract. Least-squares principles use artificial "energy" functionals to provide a Rayleigh-Ritz-like setting for the finite element method. These functionals are defined in terms of PDE's residuals and are not unique. We show that viable methods result from reconciliation of a mathematical setting dictated by the *norm-equivalence* of least-squares functionals with *practicality constraints* dictated by the algorithmic design. We identify four universal patterns that arise in this process and develop this paradigm for first-order ADN elliptic systems. Special attention is paid to the effects that each discretization pattern has on the computational and analytic properties of finite element methods, including error estimates, conditioning of the algebraic systems and the existence of efficient preconditioners.

Key Words. finite elements, least-squares, first-order elliptic systems.

1. Introduction

After a somewhat disappointing start in the early seventies[1], the use of least-squares finite elements has been steadily increasing over the last decade. A key factor for the renewed interest in such methods was the idea of their application to equivalent first-order systems rather than to the original PDE problem; see [17], [22], [18], [11] and [13]. This paid off in turning least-squares methods into a viable alternative to Galerkin finite elements, especially in fluid flow computations; see [6]–[12], [18]–[21], [23], and [27]–[29]. From a mathematical viewpoint another notion, namely the concept of *norm-equivalent* least-squares "energy" functionals emerged as a universal prerequisite for recovering fully the Rayleigh-Ritz setting. However, it was soon realized that norm-equivalence is often in conflict with practicality, even for first-order systems (see [6], [11] and [12]); and because practicality is usually the rigid constraint in the algorithmic development, norm equivalence was often neglected.

The main goal of this paper is to establish the reconciliation between practicality, as driven by algorithmic development, and norm-equivalence, as motivated by mathematical analyses, as the defining paradigm of least-squares finite element methods. The key components of this paradigm are a *continuous least-squares principle* (CLSP) which describes a mathematically well-posed, but perhaps impractical,

Received by the editors January 10, 2004.
2000 *Mathematics Subject Classification.* 65F10 (Primary), 65F30 (Secondary).
Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

[1]Early examples of least-squares methods suffered from disadvantages that seriously limited their appeal. In many cases discretization required impractical C^1 or better finite element spaces and led to algebraic problems with higher than usual condition numbers; see e.g., [3]–[4], and without efficient preconditioners.

variational setting, and an associated *discrete least-squares principle* (DLSP) which describes an algorithmically feasible setting. The relation between a CLSP and a DLSP follows four universal patterns which lead to four classes of least-squares finite element methods with distinctly different properties.

We develop this paradigm for the important class of first-order systems that are elliptic in the sense of Agmon-Douglis-Nirenberg [1]. In particular, we show that degradation of fundamental properties of least-squares methods such as condition numbers, asymptotic convergence rates, and existence of spectrally equivalent preconditioners occurs when DLSP deviates from the conforming setting induced by a given CLSP.

In what follows Ω will denote a simply connected bounded region in $\mathbb{R}^n$, $n = 2, 3$ with a sufficiently smooth boundary Γ. Throughout the paper we employ the usual notations $H^d(\Omega)$, $\|\cdot\|_d$; $d \geq 0$ for the Sobolev spaces of all functions having square integrable derivatives up to order d on Ω, and the standard Sobolev norm, respectively. As usual, $H_0^d(\Omega)$ will denote the closure of $C^\infty(\Omega)$ with respect to the norm $\|\cdot\|_d$ and $H^{-d}(\Omega)$ will denote the dual of $H_0^d(\Omega)$. The symbol S_d^h will stand for a space of continuous, piecewise polynomial functions defined with respect to a regular triangulation $\mathcal{T}_h$ of the domain Ω. It is assumed that for every $u \in H^{d+1}(\Omega)$ there exists $u^h \in S_d^h$ with

$$\|u - u^h\|_0 + h\|u - u^h\|_1 \leq Ch^{d+1}\|u\|_{d+1}. \tag{1}$$

For regular triangulations the Euclidean norm of the coefficient vector of u^h, denoted by $|\xi|$, and the L^2 norm of u^h are related by the inequality

$$C^{-1}h^M|\xi| \leq \|u^h\|_0 \leq Ch^M|\xi|\,, \tag{2}$$

where M denotes the dimension of S_d^h. We will also need the inverse inequality

$$\|u^h\|_1 \leq Ch^{-1}\|u^h\|_0 \tag{3}$$

which holds for most standard finite element spaces on regular triangulations; see [16].

2. Continuous and discrete least-squares principles

We consider boundary value problems of the form

$$\mathcal{L}(\mathbf{x}, D)\ \mathbf{u} = \mathbf{f} \ \text{ in } \Omega \quad \text{and} \quad \mathcal{R}(\mathbf{x}, D)\ \mathbf{u} = \mathbf{g} \ \text{ on } \Gamma. \tag{4}$$

Here $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ is a vector of dependent variables, $\mathcal{L}(\mathbf{x}, D) = \mathcal{L}_{ij}(\mathbf{x}, D)$, $i, j = 1, \ldots, N$ and $\mathcal{R}(\mathbf{x}, D) = \mathcal{R}_{lj}(\mathbf{x}, D)$, $l = 1, \ldots, L$, $j = 1, \ldots, N$. For simplicity, in what follows we will write $\mathcal{L}\mathbf{u}$ and $\mathcal{R}\mathbf{u}$. Concerning (4), we make the following assumption:

A.: There exist Hilbert spaces $X = X(\Omega)$, $Y = Y(\Omega)$, and $Z = Z(\Gamma)$ such that

$$C_2\|\mathbf{u}\|_X \leq \|\mathcal{L}\mathbf{u}\|_Y + \|\mathcal{R}\mathbf{u}\|_Z \leq C_1\|\mathbf{u}\|_X\,. \tag{5}$$

This relation is fundamental to least-squares methods because it defines the proper "balance" between solution energy as measured by $\|\mathbf{u}\|_X$ and data energy, as measured by $\|\mathcal{L}\mathbf{u}\|_Y + \|\mathcal{R}\mathbf{u}\|_Z$. We note that the setting determined by (5) is not, in general, unique[2].

[2]For example, if $(\mathcal{L}, \mathcal{R})$ has a complete set of homeomorphisms (5) holds on a Hilbert scale; see [24] and [25].