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THE MECHANICAL BEHAVIOR OF A POROELASTIC MEDIUM SATURATED WITH A NEWTONIAN FLUID

ZHANGXIN CHEN, STEPHEN L. LYONS, AND GUAN QIN

Abstract. In this paper we systematically derive, via the theory of homogenization, the macroscopic equations for the mechanical behavior of a deformable porous medium saturated with a Newtonian fluid. The derivation is first based on the equations of linear elasticity in the solid, the Stokes equations for the fluid, and suitable conditions at the fluid-solid interface. A detailed comparison between the equations derived here and those by Biot is given. The homogenization approach determines the form of the macroscopic constitutive relationships between variables and shows how to compute the coefficients in these relationships. The derivation is then extended to the nonlinear Navier-Stokes equations for the fluid in the deformable porous medium for the first time. A generalized Forchheimer law is obtained to take into account the nonlinear inertial effects on the flow of the Newtonian fluid through such a medium. Both quasi-static and transient flows are considered in this paper. The properties of the macroscopic coefficients are studied. The computational results show that the macroscopic equations predict well the behavior of the microscopic equations in certain reasonable test cases.

Key Words. deformable porous medium, Forchheimer law, homogenization, linear elasticity, high flow rate, Navier-Stokes equation, computational validation.

1. Introduction

We have recently employed the theory of homogenization to derive the Forchheimer law directly from the nonlinear Navier-Stokes equation in a rigid porous medium [11]. Unlike other studies based on the same approach that concluded the nonlinear correction to be cubic in velocity for an isotropic medium, our work has shown that the nonlinear correction is quadratic. In this paper we extend the techniques in [11] to a deformable porous medium.

The macroscopic mechanical behavior of a deformable porous medium has been studied by Biot [5, 6, 7] by means of an intuitive approach. Later studies have been based on mixture theory and constitutive assumptions [12]. Recent studies have utilized a group of averaging approaches [1, 9, 22] for treating Stokes flow through a periodic deformable medium. These averaging approaches have reproduced Biot's equations and shown how to calculate the coefficients in these equations. In this paper Stokes flow through a deformable medium is further examined by adapting the approach in [11]. This approach is simpler and is more direct than those in [1, 9, 22] since lower-order approximation terms are used, while higher-order terms were exploited in [1, 9, 22]. Also, the coefficients in the macroscopic equations are

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studied here in detail and a detailed comparison between the equations we obtain and those by Biot is given. Finally, we analyze the nonlinear Navier-Stokes flow in the framework of a deformable porous medium for the first time.

The approach in [11] is based on the theory of two-scale homogenization [4, 20]. The two-scale homogenization for a periodic porous medium averages the detailed microstructure of the pores and yields a set of simpler, macroscopic equations. This is achieved by a careful scaling of the microscopic equations by the ratio of two length scales associated with the microscopic and macroscopic phenomena in the periodic medium.

To compare the present theory with Biot's theory [5, 6, 7] for deformable porous medium flow, we carry out the analysis by starting with the equations of linear elasticity in the solid, the Stokes equations for the fluid, and suitable equations at the solid-fluid interface. The macroscopic equations derived here coincide with Biot's equations in the case where the scaled viscosity (see the definition in the next section) of the fluid is small. Moreover, the present theory determines the form of the macroscopic constitutive relationships between variables and shows how to compute the coefficients in these relationships. In the case where the scaled viscosity is large, we derive different differential equations and constitutive relationships. Two situations in this case are investigated. The first situation concerns the elastic single-phase behavior of the porous system, while the second concerns the viscoelastic single-phase behavior of this system.

We then extend the analysis to the nonlinear Navier-Stokes equations for the fluid. When these nonlinear equations are analyzed in the framework of a deformable medium, the situation is more complicated. It is well known that the simplest law for describing the flow of a fluid in a porous medium is the law obtained by Darcy (1856) [13]. Derived from empiricism, this law indicates a linear relationship between the fluid velocity relative to the solid and the pressure gradient. Subsequently, Dupuit (1863) [14] and Forchheimer (1901) [15] gave further empirical evidence that the linearity in Darcy's law does not hold for high rates of fluid flow and generalized this law in a nonlinear fashion (i.e., Forchheimer's law). In this paper we derive a generalized Forchheimer law for a deformable porous medium to take into account the nonlinear inertial effects on the fluid flow through such a medium.

The paper is organized as follows. In the next section we consider Stokes flow. Then, in the third section we analyze Navier-Stokes flow. The quasi-static case is considered in these two sections. Transient inertial effects are taken into account in the fourth section. In the fifth section we present a computational validation of some of the homogenized models derived. Concluding remarks are stated in the last section. The properties of the macroscopic coefficients are studied in the appendix. We end with two remarks. First, vectors and matrices will be represented by bold face variables, and the rectangular coordinates in $\Re^3$ are denoted by $x = (x_1, x_2, x_3)$ (not in bold face). Second, in this paper we focus on the systematical derivation of the macroscopic equations for the mechanical behavior of a deformable porous medium saturated with a Newtonian fluid via homogenization, the comparison of the present theory with Biot's, and the study of properties of macroscopic coefficients. A convergence proof of the homogenization approach in the present setting is beyond the scope of this paper.