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AN H^1-GALERKIN MIXED METHOD FOR SECOND ORDER HYPERBOLIC EQUATIONS

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This paper is dedicated to our beloved teacher Professor Purna Chandra Das on the occasion of his 65th birthday

Abstract. An H^1- Galerkin mixed finite element method is discussed for a class of second order hyperbolic problems. It is proved that the Galerkin approximations have the same rates of convergence as in the classical mixed method, but without LBB stability condition and quasi-uniformity requirement on the finite element mesh. Compared to the results proved for one space variable, the $L^\infty(L^2)$-estimate of the stress is not optimal with respect to the approximation property for the problems in two and three space dimensions. It is further noted that if the Raviart- Thomas spaces are used for approximating the stress, then optimal estimate in $L^\infty(L^2)$-norm is achieved using the new formulation. Finally, without restricting the approximating spaces for the stress, a modification of the method is proposed and analyzed. This confirms the findings in a single space variable and also improves upon the order of convergence of the classical mixed procedure under an extra regularity assumption on the exact solution.

Key Words. Second order wave equation, LBB condition, H^1 Galerkin mixed finite element method, semidiscrete scheme, completely discrete method, optimal error estimates.

1. Introduction

In this paper, we discuss a new mixed finite element method for the following second order hyperbolic initial and boundary value problem

$$u_{tt} - \nabla\cdot(a(x)\nabla u) + c(x)u = f(x,t), \quad (x,t)\in\Omega\times J, \tag{1.1}$$
$$u = 0, \quad (x,t)\in\partial\Omega\times J,$$
$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \; x\in\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^d, d = 2,3$ with boundary $\partial\Omega$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$ and $J = (0,T]$ with $T < \infty$. Assume that the coefficients a, c, initial functions u_0, u_1 and the forcing function f are sufficiently smooth with $a \geq a_0$ for some positive constant a_0, and $c \geq 0 \; \forall x \in \Omega$.

When our primary concern is to obtain both displacement, i.e., u and the stress that is, $\sigma = a\nabla u$, we first split (1.1) into a system of two equations and then apply classical mixed methods, see [5], [8] and [9]. However, this procedure has to satisfy the LBB-stability condition on the approximating subspaces and this restricts the choice of finite element spaces. For example, the Raviart-Thomas

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spaces of index $r \geq 1$ are usually used for the standard mixed methods. In order to avoid using LBB-stability condition, we introduce, in this paper, an alternate mixed finite element procedure. The proposed method is a non-symmetric version of least square method and we shall call it as H^1- Galerkin mixed finite element procedure. It takes advantage of the least-square method and yields a better rate of convergence for the stress than the conventional use of linear elements.

In recent years, substantial progress has been made in the least-square mixed methods applied mainly to the elliptic equations, see [4], [7], [10]–[11], [15]–[18], and references, there in. These procedures that circumvent the LBB stability conditions are considered as alternatives to the classical mixed formulations. So far, there has been hardly any analysis for the least square methods applied to parabolic and second order hyperbolic initial and boundary value problems. In an attempt to extend least-square mixed methods to parabolic equations, one of the authors [12] has introduced an H^1-Galerkin mixed procedure ,i.e., a non-symmetric version of least square method and has derived optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms. For more applications of this alternate mixed formulation, see [13]–[14].

In the present article, the proposed mixed method is applied to a system consisting of displacement u and stress σ. The approximating finite element spaces V_h and $\mathbf{W_h}$ are allowed to be of differing polynomial degrees. Hence, estimations have been obtained which distinguish the better approximation properties of V_h and $\mathbf{W_h}$. Compared to classical mixed methods, the present method is not subject to LBB stability condition. While in classical method, the $L^\infty(L^2)$-estimate of the stress is suboptimal, optimal estimate is derived for the problems in one space dimension using this new mixed formulation. It is noted that if the finite element spaces for approximating the stress are of Raviart-Thomas type, then optimal estimates are achieved for the stress. Finally, without restricting the finite dimensional spaces, a modification of the H^1-Galerkin method is proposed and analyzed. Although an extra regularity condition is required on the exact solution, yet an optimal order of convergence with respect to the approximation property for the stress in $L^\infty(L^2)$-norm is established (see, Remarks 2.1, 3.1 and Section 4 below). Moreover, it is noted that the quasi-uniformity condition is not imposed on the finite element mesh for the error estimates in L^2 and H^1-norms.

The layout of the paper is as follows. In Section 2, a second order hyperbolic equation in a single space variable is considered and optimal error estimates are discussed for the semidiscrete case. In two- and three space dimensions, a similar analysis is carried out in Section 3. Moreover, the rates of convergence obtained coincide with those in [5], [8] using classical mixed method. But compared to one dimensional case, the $L^\infty(L^2)$- estimate of the stress in this section is not optimal. It is, further, noted that if the Raviart-Thomas finite element spaces are used for approximating the stress, then we obtain optimal estimate. In Section 4, a modification of H^1- Galerkin mixed finite element method is examined without resricting the approximating spaces for the stress and semidiscrete error estimates are established. In Section 5, a completely discrete scheme is briefly described and *a priori* error bounds are derived only for the modified H^1- Galerkin mixed method.

Throughout this paper, C will denote a generic positive constant which does not depend on the spatial mesh parameter h and time discretization parameter Δt.