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CONVERGENCE AND STABILITY OF EXPLICIT/IMPLICIT SCHEMES FOR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. In this paper an explicit/implicit schemes for parabolic equations with discontinuous coefficients is analyzed. We show that the error of the solution in L^∞ norm and the error of the discrete flux in L^2 norm are in order $O(\tau + h^2)$ and $O(\tau + h^{\frac{3}{2}})$, respectively and the scheme is stable under some weaker conditions, while the difference scheme has the truncation error $O(1)$ at the neighboring points of the discontinuity of the coefficient. Numerical experiments, which are given for both linear and nonlinear problems, show that our theoretical estimates are optimal in some sense. The comparison with some classical scheme is presented.

Key Words. Domain decomposition, parabolic equations, discontinuous coefficient, parallel difference schemes, convergence.

1. Introduction

Multi–material systems are considered in many physical applications, e.g., the heat conduction procedure. When there are several materials in contact with each other, the conductivity coefficient can be varying, and discontinuous on the interface of the contact. Sometimes the conductivity coefficients differ in quantity order very much from one another.

Consider the initial–boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad 0 < t \le T, \tag{1}$$

$$u(0,t) = \beta_1(t), \quad u(1,t) = \beta_2(t), \quad 0 < t \le T, \tag{2}$$

$$u(x,0) = \alpha(x), \quad 0 < x < 1, \tag{3}$$

where the positive function $a(x)$ is the conductive coefficient, and $f(x,t)$ is the source term. We suppose that the functions are piecewise–smooth with discontinuity of first kind at $x = \xi$, where $\xi \in (0,1)$ be a fixed point. Denote $\Omega^- = \{0 \le x \le \xi, 0 \le t \le T\}$, $\Omega^+ = \{\xi \le x \le 1, 0 \le t \le T\}$ and $l = \Omega^- \cap \Omega^+$. The value of a function at $x = \xi$ is denoted by subscript ξ, and the left and right limit at ξ are denoted by subscript L and R respectively. For example, define

$$u_\xi(t) = u(\xi,t), \quad u_L(t) = \lim_{x\to\xi^-} u(x,t), \quad u_R(t) = \lim_{x\to\xi^+} u(x,t).$$

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Assume the following conditions hold.

(I) There are positive constants $\sigma, \sigma_L, \sigma_R$ and C such that

$$a(x) \geq \sigma, \ \ \forall x \in [0,1],$$

$$\sup_{0\leq x\leq \xi} a(x) = \sigma_L, \quad \sup_{\xi\leq x\leq 1} a(x) = \sigma_R, \quad \sup_{(x,t)\in\Omega^-\cup\Omega^+} |f(x,t)| \leq C.$$

(II) $\alpha(x)$, $a(x)$ and $f(x,t)$ are smooth on Ω^- and Ω^+ respectively, but have discontinuity of the first kind on l. And there holds $a_L \frac{\partial \alpha_L}{\partial x} = a_R \frac{\partial \alpha_R}{\partial x}$.

(III) Let $\beta_1(t)$ and $\beta_2(t)$ be smooth functions for $t \in [0,T]$, and the consistent conditions hold, e.g., $\alpha(0) = \beta_1(0)$, $\alpha(1) = \beta_2(0)$.

Then, it is well known (*e.g.*, see [7,8]) that (1)–(3) has an unique weak solution $u = u(x,t)$, which is smooth on Ω^- and Ω^+ respectively, and satisfy the joint condition $u_\xi = u_L = u_R$ and $K_\xi = K_L = K_R$, where K is the flux defined by $K = K(x,t) = a(x)\frac{\partial u}{\partial x}$.

There has been numerous work on numerical solution of the initial-boundary problem (1)–(3). The difficulty lies on the discontinuity of material coefficient. It has been proved in [11,12] that truncation errors for many finite difference schemes are the order $O(1)$ for such discontinuous problems. Samarskii [12,13] studied the classical θ-scheme

$$\partial_t U_j^n = \partial_x (a_{j-\frac{1}{2}} \partial_{\bar{x}} U_j^{n+\theta}), \ \ j = 1, \cdots, J-1,$$

where $U_j^{n+\theta} = \theta U_j^{n+1} + (1-\theta) U_j^n$. By an energy method, he proved for $\frac{1}{2} \leq \theta \leq 1$ that

$$\|U^n - u^n\|_\infty \leq \begin{cases} C(\tau + h), & \text{if } \ \theta = 1, \\ C(\tau + h^{\frac{1}{2}}), & \text{if } \ \frac{1}{2} < \theta < 1, \\ C(\tau^2 + h^{\frac{1}{2}}), & \text{if } \ \theta = \frac{1}{2}. \end{cases} .$$

For $\frac{1}{2} > \theta > 0$, no similar convergence result for the θ-scheme was obtained. To attain a higher rate of convergence, a modified scheme was proposed in [12,13] by using the harmonic mean over intervals of length h, $a_h(x) = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} a(x+sh)^{-1} ds\right)^{-1}$, and replacing a by a_h in the θ–scheme to obtain

$$\partial_t U_j^n = \partial_x (a_{h,j-\frac{1}{2}} \partial_{\bar{x}} U_j^{n+\theta}), \ \ j = 1, \cdots, J-1.$$

For this modified scheme it was proved that

$$\|U^n - u^n\|_\infty \leq \begin{cases} C(\tau + h^2), & \text{if } \ \theta = 1, \\ C(\tau + h^{\frac{3}{2}}), & \text{if } \ \frac{1}{2} < \theta < 1, \\ C(\tau^2 + h^{\frac{3}{2}}), & \text{if } \ \theta = \frac{1}{2}. \end{cases}$$

An alternative approach is the so-called immersed method, which has been developed for solving elliptic interface problems with finite difference approximations [6] and with finite element approximations [4,10]. The main idea in the immersed type methods is to use the interface conditions in those interface elements. In the recent work [10], an immersed finite element space is introduced. The IFE space is nonconforming and its partition is independent of the interface.

There are two types of schemes for time-dependent problems in general, implicit and explicit schemes. The former has no restriction on its time stepping. But in each time step one has to solve a global system of equations. The implementation on parallel computers is not straightforward due to its global nature. The latter is easy to program and implement on parallel computers. However, it suffers the severely restricted time stepsize from stability requirement. The classical θ-scheme is explicit for $0 < \theta < 1/2$ and implicit for $1/2 \leq \theta \leq 1$. Recently, a so-called