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AMERICAN PUT OPTIONS ON ZERO-COUPON BONDS AND A PARABOLIC FREE BOUNDARY PROBLEM

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Abstract. In this paper we study American put options on zero-coupon bonds under the CIR model of short interest rates. The uniqueness of the optimal exercise boundary and the solution existence and uniqueness of a degenerate parabolic free boundary problem are established. Numerical examples are also presented to confirm theoretical results.

Key Words. American put option, zero-coupon bond, optimal exercise boundary, free boundary problem, uniqueness, existence

1. Introduction

In this paper, we shall study American put options on zero-coupon bonds. Since bonds and their options are financial derivatives of interest rates, we need term structure models of interest rates to determine the rational prices of these financial products. In those models, the short rate of interest is considered to be a random process governed by a stochastic differential equation. Here we adopt the CIR model developed by Cox, Ingersoll and Ross in 1985 [8]. The prominent feature of this model is that the short interest rate is never negative. Indeed, the stochastic process $r(t)$ of the short interest rate under the CIR model follows the square-root dynamics:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad t > 0, \tag{1}$$

where $W(t)$ is a standard Brownian motion under the risk-neutral measure Q, κ is the speed of adjustment, θ is the long-term value of interest rate, and σ is a positive constant. It can be shown that $r(t)$ is always positive when $\kappa\theta/\sigma^2 \geq 1/2$ and that $r(t)$ can reach zero when $\kappa\theta/\sigma^2 < 1/2$.

Since the American option can be exercised at any time up to its expiration date, there is an optimal exercise boundary. The optimal exercise boundary will divide the whole domain into two regions. It is optimal to exercise the option in one region but the option should be kept in the other region. American option problems can be treated further by optimal stopping problems and by parabolic free boundary value problems. While there have been extensive studies on American stock options, American bond options have not been paid much attention in theoretical analysis. We refer the interested reader to [7], [9], [15], [19]) and references cited therein in this aspect. In this paper we shall show that there is a unique optimal exercise boundary and the corresponding free boundary problem has a unique weak solution.

The outline of the paper is as follows. As in [12] for American stock put options, we use the optimal stopping problem formulation to investigate properties of

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American put options on zero-coupon bonds in Section 2. Especially, we show the existence and uniqueness of the optimal exercise boundary. In section 3, we study the parabolic free boundary problem by variational method. The difficulty is that the partial differential operator is degenerate. The free boundary problem may be investigated by using weighted Sobolev spaces in the light of the formulation for the finite volume methods in [1] and [3]. With appropriate variable transforms, we are able to remove the degenerate factor and then propose a variational formulation with a coercive bilinear form in the usual Sobolev space. The solution uniqueness follows from the coercivity of the bilinear form. The solution existence is established by considering limit of a sequence of solutions for nonlinear variational problems of the parabolic type. In Section 4, numerical results are presented to confirm our theoretical results.

2. American put option and its optimal exercise boundary

Consider the American put option with exercise price $\$K$ and expiry date T, which is written on a zero-coupon bond with face value \$1 (without loss of generality) and maturity date T^* ($> T$). Recall that American contingent claims can be formulated as optimal stopping problems (see [5], [14] and references cited therein). Denote by $p(r,t)$ the price of the put when $r(t) = r$ at time t. Then (see [7])

$$p(r,t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} E\left[\exp\left(-\int_t^\tau r(s)ds \right) g(r(\tau),\tau) \middle| \mathcal{F}_t \right], \tag{2}$$

where $\{\mathcal{F}_t\}$ is the filtration generated by $W(t)$, $\mathcal{T}_{[t,T]}$ is the set of all stopping times assuming values in $[t,T]$, $g(r,t) = (K - B(r,t;T^*))^+$ is the payoff of the put, $z^+ = \max(z,0)$, and $B(r,t;T^*)$ is the bond price given by

$$B(r,t;T^*) = E\left[\exp\left(-\int_t^{T^*} r(s)ds \right) \middle| \mathcal{F}_t \right]. \tag{3}$$

In [8], the explicit expression of $B(r,t;T^*)$ was found as follows:

$$B(r,t;T^*) = A(T^* - t)e^{-C(T^*-t)r}$$

where $A(t)$ is a smooth and strictly decreasing function, $C(t)$ is a smooth and strictly increasing function and $A(0) = 1$, $C(0) = 0$. Therefore, $B(r,t;T^*)$ is an increasing function of t and a decreasing function of r, which is as expected in practice.

It should be pointed out that the exercise price K must be strictly less than $B(0,T;T^*) = A(T^* - T)$ which is the maximum of bond price $B(r,t;T^*)$ on $[0,\infty) \times [0,T]$. Otherwise, the exercise would be never optimal (see [8] for American call options). In fact, if $K \geq B(0,T;T^*)$, then

$$K \geq B(r,t;T^*), \ r \geq 0, \ 0 \leq t \leq T.$$

Hence it follows from (2) and (3) that