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# CONVERGENCE OF THE FINITE VOLUME METHOD FOR STOCHASTIC HYPERBOLIC SCALAR CONSERVATION LAWS: A PROOF BY TRUNCATION ON THE SAMPLE-TIME SPACE

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**Abstract.** We prove the almost sure convergence of the explicit-in-time Finite Volume Method with monotone fluxes towards the unique solution of the scalar hyperbolic balance law with locally Lipschitz continuous flux and additive noise driven by a cylindrical Wiener process. We use the standard CFL condition and a martingale exponential inequality on sets whose probabilities are converging towards one. Then, with the help of stopping times on those sets, we apply theorems of convergence for approximate kinetic solutions of balance laws with stochastic forcing.

**Key words.** Finite volume method, stochastic balance law, kinetic formulation.

## Contents



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## 1. Introduction

**Stochastic hyperbolic scalar balance law.** Let $T > 0$ be a finite time and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))_{t\in[0,T]}$ be a stochastic basis. Consider the hyperbolic scalar balance law with stochastic forcing

$$du(x,t) + \operatorname{div}_x(A(u(x,t)))dt = \Phi dW(t), \quad x \in \mathbb{T}^N, t \in (0,T). \tag{1}$$

Equation (1) is periodic in the space variable $x \in \mathbb{T}^N$, where $\mathbb{T}^N$ is the $N$-dimensional torus.

**Assumption 1.1.** The flux function $A$ in (1) is supposed to be of class $C^2$: $A \in C^2(\mathbb{R};\mathbb{R}^N)$. We assume that $A$ and its derivatives have at most polynomial growth. We denote its first derivative $A' =: a$. Without loss of generality, we assume that $A(0) = 0$.

**Assumption 1.2.** The right-hand side of (1) is a stochastic increment in infinite dimension. It is defined as follows (see [8] for the general theory): $t \in [0,T] \mapsto W(t)$ is a cylindrical Wiener process, that is $\forall t \in [0,T], W(t) = \sum_{k\geq 1} \beta_k(t)e_k$, where the coefficients $\beta_k$ are independent standard Brownian motions and $(e_k)_{k\geq 1}$ is an orthonormal basis of the separable Hilbert space $H$. Denoting $L_2(H,\mathbb{R})$ the set of Hilbert-Schmidt operators from $H$ to the space of real numbers $\mathbb{R}$, we assume that

$$\Phi \in L_2(H,\mathbb{R}). \tag{2}$$

**The Cauchy Problems**. Let us quote the main results: In [14], E, Khanin, Mazin, Sinai proved uniqueness and existence of the solution of the stochastic Burgers Equation with additive noise carried by a cylindrical Wiener process. They used a periodic solution in space dimension one $x \in \mathbb{T}$ in order to prove the existence of an invariant measure. In [22], Kim proved uniqueness and existence of the solution for a more general non-linear flux, with the space variable $x \in \mathbb{R}$ and a real Brownian motion. In [16], Feng and Nualart proved uniqueness of a solution in space dimension $N \in \mathbb{N}^*$, while existence was proved only in space dimension one. They used for the first time a multiplicative noise. The existence of a solution in space dimension $N \in \mathbb{N}^*$ was proved later by Chen, Ding, Karlsen in [7]. In [5], Bauzet, Vallet and Wittbold proved uniqueness and existence of the solution for a non-linear flux, with the space variable $x \in \mathbb{R}^N$ and a multiplicative noise driven by a real Brownian motion. In [9], Debussche and Vovelle proved uniqueness and existence of the solution for a non-linear flux and a multiplicative noise driven by a cylindrical Wiener process. Their solution is periodic in space: $x \in \mathbb{T}^N$. All the previous solutions are entropic solutions defined by Kruzkov in [25]. While similar, slight differences on assumptions or formulations always exist. For exemple [5] is following the formulation of Di Perna [10] with the measure-valued solution, while [9] is following the kinetic formulation of Lions, Perthame, Tadmor [27]. They all first proved uniqueness, then existence via the approximation given by the stochastic parabolic equation. In [12], we followed the works of [9] by using a kinetic formulation, a multiplicative noise driven by a cylindrical Wiener process, and defining a periodic solution for the space variable $x \in \mathbb{T}^N$. We proved uniqueness of a solution, and a general framework for convergence of approximate solutions towards the unique solution.

**Approximations by the Finite Volume Method.** Let us quote the book of Eymard, Gallouët, Herbin [15] for the general theory, the courses of Vovelle [40] for a quick entrance in the theory, both for the deterministic case, and the recent work of [6] where the Finite Volume Method is used to approximate the invariant measure of a viscous balance law with stochastic forcing.
For the approximation of hyperbolic scalar conservation laws with stochastic source term by the Finite Volume Method, fews results exists:

- In space dimension 1, with strongly monotone fluxes, [24] proved the convergence of a semi-discrete Finite Volume Method towards the solution defined in [16]
- In space dimension $N \geq 1$, [3] proved the convergence of a fully discrete flux-splitting Finite Volume Method towards the solution defined in [5]. [18] proved it for stochastic source term in infinite dimensions.
- In space dimension $N \geq 1$, with monotone fluxes, [4] proved the convergence of a fully discrete Finite Volume Method towards the solution defined in [5]. The convergence was proved in [2] with a more general flux function.
- In space dimension $N \geq 1$, with monotone fluxes, [13] proved the convergence of a fully discrete Finite Volume Method towards the solution defined in [12] in the case of compactly supported multiplicative noise.

**Some other numerical approximations of stochastic hyperbolic scalar conservation laws.** In [31], Mishra and Schwab proved the convergence of approximations given by a multi-level Monte-Carlo Finite Volume Method for scalar hyperbolic conservation laws with random initial datum. In [32], with Risebro and Tokareva, they generalized the result of convergence for conservation laws with random flux functions. In [19], Hofmanovà proved the convergence of a Bhatnagar-Gross-Krook approximation towards the solution defined in [9]. She generalized the result of convergence in [20] for conservation laws with rough flux and space dependent multiplicative noise. In [26], Li, Shu and Tang studied the approximation given by a discontinuous Galerkin method in space dimension 1, for conservation laws with semi-linear and non-linear flux functions, driven by multiplicative noise. The stochastic homogeneous case was studied in space dimension 1 with a linear flux function by Jin and Ma in [21]. In [30], Meyer, Rohde and Giesselmann studied the error estimate of an approximation given by stochastic Galerkin methods for scalar hyperbolic balance laws with a random source term, in space dimension 1. In [23], Koley, Majee and Vallet proved the convergence of approximations generated by a finite difference scheme for conservation laws driven by Lévy noise. In [17], Fjordholm, Karlsen and Pang proved the convergence of approximations given by finite difference schemes for stochastic transport equations with gradient noise.

**Some comments about the assumptions.** In [13], we used a stronger constraint on $\Phi$, that is $\Phi$ compactly supported. The aim was to do as if $A$ was globally Lipschitz continuous, in order to have a CFL condition for the Finite Volume Method. In [3] and [4], Bauzet, Charrier, Gallouët used a Lipshitz continuous flux with the same aim. Here, we still take an assumption stronger than in the continuous case (see [12]): we take an additive noise with the coefficient $\Phi \in L_2(H, \mathbb{R})$ and we keep the flux locally Lipschitz continuous. It is important to understand that, it is the first proof of convergence of the Finite Volume Method for such a stochastic balance law with locally Lipschitz continues flux and a source which is not compactly supported. The proof is also interesting in itself: To have the CFL condition of the Finite Volume Method, we will work on a subset of $\Omega$ where the approximate solutions remain in a compact of $\mathbb{R}$. The probability of the complement of that subset can be made as small as wished. In this way, we can work as if the flux $A$ was globally Lipschitz continuous, at least on a subset of $\Omega$. It is a type of truncation on the sample-time space because, to keep the random variable in that subset of $\Omega$, the time variable has to remain smaller than the value of a particular stopping time, defined with an exponential martingale inequality. We hope that this result and this method (truncation on the sample-time space $\Omega \times [0, T]$) is the first step to prove the convergence of the Finite Volume Method in the case of multiplicative noise with bounded multiplier using comparisons of diffusion processes.

**Kinetic formulation.** To prove the convergence of the Finite Volume Method with monotone fluxes, we will use the companion paper [12] and a kinetic formulation of the Finite Volume Method. The subject of [12] is the convergence of approximations of (1) in the context of the kinetic formulation of scalar balance laws. Such kinetic formulations have been developed in [27, 28, 29, 34, 35] for deterministic conservation laws. In [29], a kinetic formulation of Finite Volume E-schemes is given (and applied in particular to obtain sharp CFL criteria). Here we modify the sequence of generalized approximate kinetic solutions in this way: we keep the sequence of [13] on a subspace of $\Omega$ whose probability tends towards one when a parameter $\lambda$ tends towards infinity, by multiplying by the indicator function of a time interval ended by a stopping time. In that way, verifying the assumptions of convergence given by theorem 2.3 which is a generalization of theorem 4.15 from [12], we get the convergence in probability of the approximate solution given by the Finite Volume Method, then its almost sure convergence.

**Plan of the paper.** The plan of the paper is the following one. In the preliminary section 2, following [12], we give the definition of a solution of (1), we give a more general definition of the so-called generalized solution, we generalize both to have the definitions of a solution or a generalized solution of (1) up to a stopping time. We give the result of uniqueness of a solution and reduction of a generalized solution to a solution from [12] that we generalized for solution and generalized solution up to a stopping time. We give the corresponding result of convergence of approximate solutions towards the solution of (1) up to a stopping time. In Section 3, we describe the approximations of (1) given by the Finite Volume Method. In Section 4 we establish the kinetic formulation of the scheme. We find a subspace of $\Omega$ on which the numerical values given by the scheme are bounded, in order to define the proper CFL condition of the scheme. This subspace is found by the use of an exponential martingale inequality. In section 5, we define the sequence of generalized approximate solutions up to a stopping time $(f_\delta)$ and the associated sequence of Young measures $(\nu^\delta)$ and sequence of kinetic measures $(m^\delta)$. We prove the tightness of $(\nu^\delta)$ and $(m^\delta)$. In section 6, by the use of a numerical kinetic equation, we prove that $(f_\delta)$ verify an approximate kinetic time-continuous equation. In section 7, we conclude by the convergence in probability of the approximate solution given by the Finite Volume Method towards the unique solution of (1). Then, we prove the almost sure convergence of the Finite Volume Method, as a consequence of the convergence in probability and the uniqueness of the solution of (1). It is the main result of the paper. In section 8, we discuss possible ways to improve our main result, by a more general initial data, a more general noise or by a stronger mode of convergence.

## 2. Generalized solutions, approximate solutions

The object of this section is to re-write several results concerning the solutions of the Cauchy Problem associated with (1) and their approximations proved in [12] with slight modifications in order to be used with stopping times.

### 2.1. Solutions.

**Definition 2.1** (Random measure)**.** Let $\mathcal{M}_b^+\left(\mathbb{T}^N \times [0,T] \times \mathbb{R}\right)$ be the set of bounded Borel non-negative measures. If $m$ is a map from $\Omega$ to $\mathcal{M}_b^+\left(\mathbb{T}^N \times [0,T] \times \mathbb{R}\right)$ such that, for each continuous and bounded function $\phi$ on $\mathbb{T}^N \times [0,T] \times \mathbb{R}$, $\langle m, \phi\rangle$ is a random variable, then we say that $m$ is a random measure on $\mathbb{T}^N \times [0,T] \times \mathbb{R}$.

**Definition 2.2** (Solution)**.** Assume that assumptions 1.1, 1.2 are satisfied. Let $u_0 \in L^\infty(\mathbb{T}^N)$. A $L^1(\mathbb{T}^N)$-valued stochastic process $(u(t))_{t\in[0,T]}$ is said to be a solution of (1) with initial datum $u_0$ if $u$ and $\mathbf{1}_{u>\xi}$ have the following properties:

(1) $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0,T] \times \Omega)$,

(2) for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, almost surely, $t \mapsto \displaystyle\int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi)\mathbf{1}_{u(x,t)>\xi}\, dxd\xi$ is càdlàg,

(3) for all $p \in [1,+\infty)$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u(t)\|^p_{L^p(\mathbb{T}^N)}\right) \leq C_p, \tag{3}$$

(4) there exists a random measure $m$ defined in 2.1, verifying

$$\mathbb{E}m(\mathbb{T}^N \times [0,T] \times \mathbb{R}) < +\infty, \tag{4}$$

such that for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, almost surely, for all $t \in [0,T]$

$$\begin{aligned} &\int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi)\mathbf{1}_{u(x,t)>\xi}\, dxd\xi \\ &= \int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi)\mathbf{1}_{u_0(x)>\xi}\, dxd\xi + \int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} a(\xi)\cdot\nabla\varphi(x,\xi)\mathbf{1}_{u(x,s)>\xi}\, dxd\xi ds \\ &\quad + \int_0^t \int_{\mathbb{T}^N} \varphi(x,u(x,s))dx\, \Phi\, dW(s) \\ &\quad + \frac{1}{2}\int_0^t \int_{\mathbb{T}^N} \partial_\xi\varphi(x,u(x,s))\, \|\Phi\|^2_{L_2(H,\mathbb{R})}\, dxds - m(\partial_\xi\varphi)([0,t]), \end{aligned} \tag{5}$$

where $a(\xi) := A'(\xi)$.

*Remark* 2.1. In item 1, the index $\mathcal{P}$ in $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0,T] \times \Omega)$ means that $u$ is predictable (see [12, Section 2.1.1]).

**Definition 2.3** (solution up to a stopping time)**.** If we fix a predictable stopping time $\tau : \Omega \to [0,T]$, and if we replace in items 1, 2, 3, 4 of definition 2.2 $u(x,t)$ by $u(x,t\wedge\tau)$, then the process $(u(t\wedge\tau))_{t\in[0,T]}$ is called solution of (1) with initial datum $u_0$ up to the stopping time $\tau$.

*Remark* 2.2. A solution of (1) with initial datum $u_0$ is a solution of (1) with initial datum $u_0$ up to any predictable stopping time $\tau : \Omega \to [0,T]$.

## 2.2. Generalized solutions.

**Definition 2.4** (Young measure)**.** Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$ endowed with the Borel $\sigma$-algebra. We say that a map $\nu\colon X \to \mathcal{P}_1(\mathbb{R})$ is a Young measure on $X$ if, for all $\phi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\phi)$ from $X$ to $\mathbb{R}$ is measurable. We say that a Young measure $\nu$ vanishes at infinity if, for every $p \geq 1$,

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) < +\infty. \tag{6}$$

**Definition 2.5** (Kinetic function)**.** Let $(X, \mathcal{A}, \lambda)$ be a finite measure space. A measurable function $f\colon X \times \mathbb{R} \to [0,1]$ is said to be a kinetic function if there exists a Young measure $\nu$ on $X$ that vanishes at infinity such that, for $\lambda$-a.e. $z \in X$, for all $\xi \in \mathbb{R}$,

$$f(z,\xi) = \nu_z(\xi, +\infty).$$

Moreover, if there exists a measurable function $u\colon X \to \mathbb{R}$ such that $f(z,\xi) = \mathbf{1}_{u(z)>\xi}$ a.e., or, equivalently, $\nu_z(d\xi) = \delta_{u(z)}(d\xi)$ for a.e. $z \in X$, $f$ is called an equilibrium.

**Definition 2.6** (Generalized solution)**.** Assume that assumptions 1.1, 1.2 are satisfied. Let $f_0\colon \mathbb{T}^N \times \mathbb{R} \to [0,1]$ be a kinetic function, an $L^\infty(\mathbb{T}^N \times \mathbb{R}; [0,1])$-valued process $(f(t))_{t\in[0,T]}$ is said to be a generalized solution of (1) with initial datum $f_0$ if $f(t)$ and $\nu_t := -\partial_\xi f(t)$ have the following properties:

(1) Almost surely, for all $t \in [0,T]$, $f(.,t,.,\omega)$ is a kinetic function, and, for all $R > 0$, $f \in L^1_{\mathcal{P}}(\mathbb{T}^N \times (0,T) \times (-R,R) \times \Omega)$,

(2) $\forall \varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, a.s., $t \mapsto \displaystyle\int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi) f(x,t,\xi) dx d\xi$ is càdlàg on $[0,T]$,

(3) for all $p \in [1, +\infty)$, there exists $C_p \in \mathbb{R}_+^*$ such that

$$\mathbb{E}\left( \sup_{t\in[0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi) dx \right) \leq C_p, \tag{7}$$

(4) there exists a random measure $m$ verifying

$$\mathbb{E} m(\mathbb{T}^N \times [0,T] \times \mathbb{R}) < +\infty, \tag{8}$$

such that for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, almost surely, for all $t \in [0,T]$

$$\begin{aligned} &\int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi) f(x,t,\xi) dx d\xi \\ &= \int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi) f_0(x,\xi) dx d\xi + \int_0^t \int_{\mathbb{T}^N\times\mathbb{R}} a(\xi)\cdot\nabla\varphi(x,\xi) f(x,s,\xi) dx d\xi ds \\ &+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \varphi(x,\xi) d\nu_{x,s}(\xi) dx \ \Phi \ dW(s) \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \, \|\Phi\|^2_{L_2(H,\mathbb{R})} \, d\nu_{x,s}(\xi) dx ds - m(\partial_\xi\varphi)([0,t]). \end{aligned} \tag{9}$$

**Definition 2.7** (generalized solution up to a stopping time)**.** If we fix a predictable stopping time $\tau : \Omega \to [0,T]$, and if we replace in items 1 , 2 , 3 , 4 of definition 2.6 $f(x,t,\xi)$ by $f(x, t\wedge\tau, \xi)$, then the process $(f(t\wedge\tau))_{t\in[0,T]}$ is called generalized solution of (1) with initial datum $f_0$ up to the stopping time $\tau$.

*Remark* 2.3. A generalized solution of (1) with initial datum $f_0$ is a generalized solution of (1) with initial datum $f_0$ up to any predictable stopping time $\tau : \Omega \to [0,T]$.

The following statement is Theorem 3.2 in [12].

**Theorem 2.1** (Uniqueness, Reduction)**.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume assumptions 1.1, 1.2, then we have the following results:*

(1) *there is at most one solution $u$ with initial datum $u_0$ of (1).*
(2) *If $f$ is a generalized solution of (1) with initial datum $f_0 = \mathbf{1}_{u_0>\xi}$, then there exists a solution $u$ of (1) with initial datum $u_0$ such that $f(x,t,\xi) = \mathbf{1}_{u(x,t)>\xi}$ a.s., for a.e. $(x,t,\xi)$.*
(3) *If $u_1$, $u_2$ are two solutions of (1) associated with the initial data $u_{1,0}$, $u_{2,0} \in L^\infty(\mathbb{T}^N)$ respectively, then for all $t \in [0,T]$:*

$$\mathbb{E}\|(u_1(t) - u_2(t))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|(u_{1,0} - u_{2,0})^+\|_{L^1(\mathbb{T}^N)}. \tag{10}$$

*This implies the $L^1$-contraction property, and comparison principle for solutions.*

*Remark* 2.4. We have also uniqueness of a solution of (1) up to a stopping time and we can reduce a generalized solution up to a stopping time to a solution up to the same stopping time, using a $L^1$-contraction property.

**Theorem 2.2.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$ and $\tau : \Omega \to [0,T]$ a predictable stopping time. Assume assumptions 1.1, 1.2, then we have the following results:*

(1) *there is at most one solution $u$ of (1) with initial datum $u_0$ up to the stopping time $\tau$.*
(2) *If $f$ is a generalized solution of (1) up to the stopping time $\tau$ with initial datum $f_0 = \mathbf{1}_{u_0>\xi}$, then there exists a solution $u$ of (1) up to the stopping time $\tau$, with initial datum $u_0$ such that $f(x,t\wedge\tau,\xi) = \mathbf{1}_{u(x,t\wedge\tau)>\xi}$ a.s., for a.e. $(x,t,\xi)$.*
(3) *If $u_1$, $u_2$ are two solutions of (1) up to the stopping time $\tau$, associated with the initial data $u_{1,0}$, $u_{2,0} \in L^\infty(\mathbb{T}^N)$ respectively, then for all $t \in [0,T]$:*

$$\mathbb{E}\|(u_1(t\wedge\tau) - u_2(t\wedge\tau))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|(u_{1,0} - u_{2,0})^+\|_{L^1(\mathbb{T}^N)}. \tag{11}$$

*This implies the $L^1$-contraction property, and comparison principle for solutions.*

*Proof.* It is the same proof as the one of theorem 2.3 in [12], replacing $t$ by $t \wedge \tau$. It works because propositions 2.8, 2.10, 2.11 and proposition 3.1 in [12] can also be written replacing $t$ by $t \wedge \tau$. □

**2.3. Approximate solutions.** In [12], we gave a general framework and sufficient conditions for the convergence of sequences of approximations of (1). For the convergence of our Finite Volume Method, we need a definition of approximate generalized solutions which is weaker, that is more general than the one given in [12], that is for which the time evolution of approximate generalized solutions is stopped by a stopping time. Technically, the aim is to use a localisation procedure to have locally bounded sequences of approximations of (1) given by the Finite Volume Method. The following definition is in fact the definition 4.1 of [12] generalized for the solution of (1) up to a predictable stopping time.

**Definition 2.8** (Approximate generalized solutions up to a stopping time)**.** Assume assumptions 1.1 and 1.2. For each $n \in \mathbb{N}$, let $f_0^n\colon \mathbb{T}^N \times \mathbb{R} \to [0,1]$ be a kinetic function. Let $\tau : \Omega \to [0,T]$ be a predictable stopping time. Let $\big((f^n(t))_{t\in[0,T]}\big)_{n\in\mathbb{N}}$ be a sequence of $L^\infty(\mathbb{T}^N \times \mathbb{R}; [0,1])$-valued processes. Assume that the functions $f^n(t)$, and the associated Young measures $\nu_t^n = -\partial_\xi f^n(t)$ are satisfying items 1, 2, 3 in definition 2.7. Assume that there exist adapted processes $\varepsilon^n(t\wedge\tau,\varphi)$ such that for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, $t \mapsto \varepsilon^n(t\wedge\tau,\varphi)$ are almost surely continuous, and the sequences $(\varepsilon^n(t\wedge\tau,\varphi))_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} |\varepsilon^n(t\wedge\tau,\varphi)| = 0 \text{ in probability.} \tag{12}$$

Assume that there exist random measures $m^n$ verifying (8), such that, for all $n$, for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, almost surely, for all $t \in [0,T]$:

$$\begin{aligned}\int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi) f^n(x,t\wedge\tau,\xi)dxd\xi = &\int_{\mathbb{T}^N\times\mathbb{R}} \varphi(x,\xi) f_0^n(x,\xi)dxd\xi + \varepsilon^n(t\wedge\tau,\varphi)\\ &+ \int_0^{t\wedge\tau}\int_{\mathbb{T}^N\times\mathbb{R}} a(\xi)\cdot\nabla\varphi(x,\xi) f^n(x,s,\xi)dxd\xi ds\\ &+ \int_0^{t\wedge\tau}\int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi(x,\xi)d\nu_{x,s}^n(\xi)dx\ \Phi\ dW(s)\\ &+ \frac{1}{2}\int_0^{t\wedge\tau}\int_{\mathbb{T}^N}\int_{\mathbb{R}} \partial_\xi\varphi(x,\xi)\,\|\Phi\|^2_{L_2(H,\mathbb{R})}\,d\nu_{x,s}^n(\xi)dxds\\ &- m^n(\partial_\xi\varphi)([0,t\wedge\tau]).\end{aligned} \tag{13}$$

Then we say that $(f^n)$ is a sequence of approximate generalized solutions of (1) with initial datum $f_0^n$ up to the stopping time $\tau$.

For such a sequence $(f^n)$ of approximate solutions of (1) up to the stopping time $\tau$, the following bounds

(1) $\forall p \in [1,+\infty), \exists C_p \in \mathbb{R}_+^*$ independent of $n$ such that $\nu^n := -\partial_\xi f^n$ satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{T}^N}\int_{\mathbb{R}} |\xi|^p d\nu_{x,t\wedge\tau}^n(\xi)dx\right] \le C_p, \tag{14}$$

(2) the measures $\mathbb{E}m^n$ satisfy the bound

$$\sup_n \mathbb{E}m^n(\mathbb{T}^N\times[0,T]\times\mathbb{R}) < +\infty, \tag{15}$$

and the following tightness condition: if $B_R^c = \{\xi\in\mathbb{R}, |\xi|\ge R\}$, then

$$\lim_{R\to+\infty}\sup_n \mathbb{E}m^n(\mathbb{T}^N\times[0,T]\times B_R^c) = 0. \tag{16}$$

are necessary to apply the martingale method developed in section 4 of [12] in order to have the following convergence result, which is the convergence theorem 4.15 of [12] generalized for the solution of (1) up to a predictable stopping time.

**Theorem 2.3.** *Let $\tau : \Omega \to [0,T]$ be a predictable stopping time, Let $(f^n)$ be a sequence of approximate generalized solutions of* (1) *with initial datum $f_0^n$ up to the stopping time $\tau$. If $(f^n)$ is satisfying* (14), (15), (16) *and $(f_0^n)$ converges to the equilibrium function $\mathbf{1}_{u_0>\xi}$ in $L^\infty(\mathbb{T}^N\times\mathbb{R})$-weak-*, where $u_0 \in L^\infty(\mathbb{T}^N)$, we have the following results:*

- *There exists a unique solution $u(x,t\wedge\tau) \in L^1_{\mathcal{P}}(\mathbb{T}^N\times[0,T]\times\Omega)$ of* (1) *with initial datum $u_0$ up to the stopping time $\tau$ as defined in 2.3.*

- *The sequence of the terms*

$$u^n(x,t\wedge\tau)=\int_{\mathbb{R}}\xi d\nu^n_{x,t\wedge\tau}(\xi)=\int_{\mathbb{R}}(f^n(x,t\wedge\tau,\xi)-\mathbf{1}_{0>\xi})\,d\xi$$

*is converging towards $u(x,t\wedge\tau)$ in the following sense: for all $p\in[1,\infty)$,*

$$\mathbb{E}\int_0^T\int_{\mathbb{T}^N}|u^n(x,t\wedge\tau)-u(x,t\wedge\tau)|^p\,dxdt\underset{n\to+\infty}{\longrightarrow}0. \tag{17}$$

*Proof.* We have to verify some points to assert that the demonstration of section 4 in [12] is still working. First, the process

$$t\mapsto\int_0^{t\wedge\tau}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\varphi(x,\xi)d\nu^n_{x,s}(\xi)dx\;\Phi\;dW(s)$$

in equation (13) is a $(\mathcal{F}_t)_{t\in[0,T]}$-martingale because for $(e_k)_{k\in\mathbb{N}^*}$ an orthonormal basis of the separable Hilbert space $H$,

$$\sum_{k=1}^{+\infty}\int_0^T\left|\int_{T^N}\int_{\mathbb{R}}\varphi(x,\xi)d\nu^n_{x,s}(\xi)dx\;\Phi(e_k)\right|^2ds<T\|\varphi\|^2_\infty\|\Phi\|^2_{L_2(H,\mathbb{R})}<+\infty \tag{18}$$

thus we can apply the corollary 3.6 of [36]. Second, the bound (14) allows to have the proposition 4.3 of [12], then the bounds (15) and (16) allow to have the proposition 4.4 of [12]. Third, we can write proposition 4.5 of [12] replacing $t$ by $t\wedge\tau$ using the same proof. Then theorem 4.6 of [12] is also true replacing $t$ by $t\wedge\tau$ using the same proof . The proof of theorem 4.15 remains the same and proves our theorem 2.3. □

*Remark* 2.5. Using a classical localization formula, that is

$$u(x,t\wedge\tau)=u(x,t)\mathbf{1}_{t<\tau}+u(x,\tau)\mathbf{1}_{\tau\le t},$$

which is also true for $u^n$ instead of $u$, we can decompose the convergence (17) into two convergences, that is $\forall p\in[1,\infty)$:

$$\mathbb{E}\int_0^T\int_{\mathbb{T}^N}\mathbf{1}_{t<\tau}\,|u^n(x,t)-u(x,t)|^p\,dxdt\underset{n\to+\infty}{\longrightarrow}0 \tag{19}$$

and

$$\mathbb{E}\int_0^T\int_{\mathbb{T}^N}\mathbf{1}_{\tau\le t}\,|u^n(x,\tau)-u(x,\tau)|^p\,dxdt\underset{n\to+\infty}{\longrightarrow}0.$$

The convergence (19) will be used to obtain in the last section the convergence in probability of the Finite Volume Method.

*Remark* 2.6. Existence (and uniqueness) of the solution of (1) (that is up to the constant stopping time $\tau=T$) is proved by [9]. Another way to prove it is also given in section 5 of [12].

In the next section, we define the numerical approximation of (1) given by the Finite Volume Method. To prove the convergence in probability of the numerical approximation towards the unique solution of (1) defined by 2.2, we have to prove that the assumptions of Theorem 2.3 are satisfied. This will be done in sections 5 and 6. Then the result of Theorem 2.3 will be used in section 7 for the convergence of the Finite Volume Method.

## 3. The Finite Volume Method

Mesh. A mesh of $\mathbb{T}^N$ is a family $\mathcal{T}_\#$ of disjoint connected open subsets $K \in (0,1)^N$ which form a partition of $(0,1)^N$ up to a negligible set. We denote by $\mathcal{T}$ the mesh

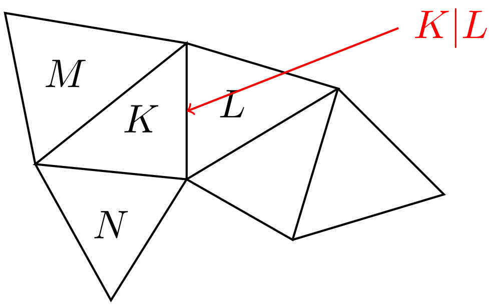
$$\{K + l : \ l \in \mathbb{Z}^N, K \in \mathcal{T}_\#\}$$

deduced on $\mathbb{R}^N$. For all distinct $K, L \in \mathcal{T}$, we assume that $\overline{K} \cap \overline{L}$ is contained in an hyperplane; the interface between $K$ and $L$ is denoted $K|L := \overline{K} \cap \overline{L}$. The set of neighbours of $K$ is

$$\mathcal{N}(K) = \{L \in \mathcal{T}; L \neq K, \ K|L \neq \emptyset\}.$$

We use also the notation

$$\partial K = \bigcup_{L \in \mathcal{N}(K)} K|L.$$




We also denote by $|K|$ the $N$-dimensional Lebesgue Measure of $K$ and by $|\partial K|$ (respectively $|K|L|$) the $(N-1)$-dimensional Haussdorff measure of $\partial K$ (respectively of $K|L$). The $(N-1)$-dimensional Haussdorff measure is normalized to coincide with the $(N-1)$-dimensional Lebesgue measure on hyperplanes, and is written $\mathcal{H}^{N-1}$ when we integrate with respect to it.
Scheme.

**Assumption 3.1.** Let $(A_{K\to L})_{K \in \mathcal{T}, L \in \mathcal{N}(K)}$ be a family of monotone, locally Lipschitz continuous numerical fluxes, consistent with the flux $A$ in (1). That is, we assume that each function $A_{K\to L} : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following properties:

- Monotonicity: $A_{K\to L}(v,w) \leq A_{K\to L}(v',w)$ for all $v$, $v'$, $w \in \mathbb{R}$ such that $v \leq v'$ and $A_{K\to L}(v,w) \geq A_{K\to L}(v,w')$ for all $v$, $w$, $w' \in \mathbb{R}$ such that $w \leq w'$.
- Local Lipschitz regularity: $\forall M \in \mathbb{R}^+$, there exists $L_A^M \in \mathbb{R}_+^*$ such that

$$|A_{K\to L}(v,w) - A_{K\to L}(v',w')| \leq |K|L| L_A^M (|v-v'| + |w-w'|), \tag{20}$$

  for all $v$, $v'$, $w$, $w' \in [-M, M] \subset \mathbb{R}$. Without loss of generality, we will assume that

$$\forall M \in \mathbb{R}_+, \quad \mathrm{Lip}(A, [-M,M]) \leq L_A^M.$$

- Consistency:

$$A_{K\to L}(v,v) = \int_{K|L} A(v) \cdot n_{K,L} \, d\mathcal{H}^{N-1}(x) = |K|L| A(v) \cdot n_{K,L}, \tag{21}$$

  for all $v \in \mathbb{R}$, where $n_{K,L}$ is the outward unit normal vector to $K$ on $K|L$.
- Conservative symmetry : for all $K, L \in \mathcal{T}$, $v, w \in \mathbb{R}$

$$A_{K\to L}(v,w) = -A_{L\to K}(w,v). \tag{22}$$

Let $0 = t_0 < ... < t_n < t_{n+1} < ... < t_{N_T} = T$ be a partition of the time interval $[0, T]$, with $N_T \in \mathbb{N}^*$. Given two discrete times $t_n$ and $t_{n+1}$, we define $\Delta t_n = t_{n+1} - t_n$ for each $n \in \{0, ..., N_T - 1\}$. Assuming assumption 1.2, and knowing $v_K^n$, an approximation of the value of the solution $u$ of (1) in the cell $K \in \mathcal{T}$ at time $t_n$, we compute $v_K^{n+1}$, the approximation of the value of $u$ in $K$ at the next time step $t_{n+1}$, by the formula

$$|K|(v_K^{n+1} - v_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} A_{K \to L}(v_K^n, v_L^n) = |K| \int_{t_n}^{t_{n+1}} \Phi \, dW(s). \tag{23}$$

Assuming that $u_0 \in L^\infty(\mathbb{T}^N)$, the initialization is given by the formula

$$v_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}. \tag{24}$$

In (23), $\Delta t_n A_{K \to L}(v_K^n, v_L^n)$ is the numerical flux at the interface $K|L$ on the range of time $[t_n, t_{n+1}]$. For practical computations, we have to define

$$X_k^{n+1} = \frac{\beta_k(t_{n+1}) - \beta_k(t_n)}{(\Delta t_n)^{1/2}} \tag{25}$$

and rewrite the equation (23) in the following way:

$$|K|(v_K^{n+1} - v_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} A_{K \to L}(v_K^n, v_L^n) = |K|(\Delta t_n)^{\frac{1}{2}} \sum_{k \geq 1} X_k^{n+1} \Phi e_k.$$

The $(X_k^{n+1})_{k \geq 1, n \in \mathbb{N}}$ are independent random variables, normally distributed with mean 0 and variance 1. Besides, for each $n \geq 1$, the sequence $(X_k^{n+1})_{k \geq 1}$ is independent of $\mathcal{F}_n$, the sigma-algebra generated by $\{X_k^{m+1}; k \geq 1, m < n\}$.

## 4. The kinetic formulation of the Finite Volume Method

**4.1. Discussion on the kinetic formulation of the Finite Volume Method.** The kinetic formulation of the Finite Volume Method has been introduced by Makridakis and Perthame in [29] for deterministic scalar conservation laws. The principle is to replace the entropic inequality by an equality, adding a kinetic entropy defect measure, which is a non-negative measure. An artificial variable $\xi \in \mathbb{R}$ called kinetic variable is also added, it comes from the constant $\xi \in \mathbb{R}$ in the Kruzkov's entropies $u \in \mathbb{R} \mapsto |u - \xi|$. We don't create a probabilistic kinetic formulation, we simply share the evolution in time of the Finite Volume Method in two artificial steps as it was already done in [13]: one deterministic step in order to use the kinetic formulation (and the results) of [29], and one probabilistic step. The state $v_K^{n+1}$ at time $t_{n+1}$ of the discrete evolution equation (23) is then reached via the two following steps: from the states $v_K^n$, the state $v_K^{n+1/2}$ is reached solving the deterministic evolution equation

$$|K|(v_K^{n+1/2} - v_K^n) + \Delta t_n \sum_{L \in \mathcal{N}(K)} A_{K \to L}(v_K^n, v_L^n) = 0, \quad \forall K \in \mathcal{T}_\#, \forall n \in \mathbb{N}, \tag{26}$$

then, from the state $v_K^{n+1/2}$, the state $v_K^{n+1}$ is reached solving the stochastic equation

$$v_K^{n+1} - v_K^{n+1/2} = \int_{t_n}^{t_{n+1}} \Phi \, dW(s). \tag{27}$$

The first step (26) corresponds the kinetic formulation

$$(28)\quad |K|\left(\mathbf{1}_{v_K^{n+1/2}>\xi}-\mathbf{1}_{v_K^n>\xi}\right)+\Delta t_n\sum_{L\in\mathcal{N}(K)}a_{K\to L}(\xi,v_K^n,v_L^n)=|K|\Delta t_n\,\partial_\xi m_K^n(\xi),$$

with discrete kinetic defect measures $m_K^n(\xi)d\xi$ verifying

$$(29)\qquad m_K^n(\xi)\geq 0,\ \forall\xi\in\mathbb{R},$$

and discrete kinetic fluxes $a_{K\to L}(\xi,v,w)$ verifying the consistency conditions

$$(30)\qquad \int_{\mathbb{R}}\left[a_{K\to L}(\xi,v,w)-|K|L|\,a(\xi).n_{K,L}\mathbf{1}_{0>\xi}\right]d\xi=A_{K\to L}(v,w),$$

$$(31)\qquad a_{K\to L}(\xi,v,v)=|K|L|\,a(\xi).n_{K,L}\mathbf{1}_{v>\xi},$$

for all $v$, $w\in\mathbb{R}$ and almost all $\xi\in\mathbb{R}$. Note that those conditions generalize the consistency conditions of Makridakis and Perthame in dimension $N=1$ (see [29]) because they use the following equation:

$$|K|\left(\mathbf{1}_{v_K^{n+1/2}>\xi}-\mathbf{1}_{0>\xi}+\mathbf{1}_{0>\xi}-\mathbf{1}_{v_K^n>\xi}\right)+\Delta t_n\sum_{L\in\mathcal{N}(K)}(a_{K\to L}(\xi,v_K^n,v_L^n)+a(\xi)\mathbf{1}_{0>\xi})=|K|\Delta t_n\,\partial_\xi m_K^n(\xi).$$

We can deduce from (27) and (28) the kinetic formulation of the whole scheme (23), that is the equation

$$(32)\quad |K|(\mathbf{1}_{v_K^{n+1}>\xi}-\mathbf{1}_{v_K^n>\xi})+\Delta t_n\sum_{L\in\mathcal{N}(K)}a_{K\to L}(\xi,v_K^n,v_L^n)=|K|\Delta t_n\,\partial_\xi m_K^n(\xi)+|K|\left[\mathbf{1}_{v_K^{n+1}>\xi}-\mathbf{1}_{v_K^{n+1/2}>\xi}\right].$$

**4.2. The CFL condition and a good truncation of $\Omega$ to make the discrete kinetic entropy defect measure non-negative.** In the next proposition, we define the discrete kinetic fluxes and the discrete kinetic defect measures to prove the existence of the kinetic formulation (28)-(29)-(30)-(31).

**Proposition 4.1** (Kinetic formulation of the Finite Volume Method)**.** *Set*

$$(33)\quad a_{K\to L}(\xi,v,w):=|K|L|\,a(\xi).n_{K,L}\mathbf{1}_{\xi<v\wedge w}+\left[\partial_2A_{K\to L}(v,\xi)\mathbf{1}_{v\leq\xi\leq w}+\partial_1A_{K\to L}(\xi,w)\mathbf{1}_{w\leq\xi\leq v}\right],$$

*then the consistency conditions* (30)-(31) *are satisfied. Defining*

$$(34)\quad m_K^n(\xi):=-\frac{1}{\Delta t_n}\left[(v_K^{n+1/2}-\xi)^+-(v_K^n-\xi)^+\right]-\frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\int_\xi^{+\infty}a_{K\to L}(\zeta,v_K^n,v_L^n)d\zeta,$$

*the equation* (28) *is satisfied. Let $\lambda\in\mathbb{R}_+^*$, adding the following CFL condition:*

$$(35)\qquad \Delta t_n\frac{|\partial K|}{|K|}L_A^{\|u_0\|_\infty+\lambda}\leq 1,\quad\forall K\in\mathcal{T},\forall n\in\{0,\cdots,N_T-1\},$$

*then the equation* (29) *is satisfied for all* $\omega \in \Omega_\lambda^b$ *the measurable subset of* $\Omega$ *defined as*

$$\Omega_\lambda^b := \Big\{\omega \in \Omega : v_K^n(\omega) \in \left[-\|u_0\|_{L^\infty(\mathbb{T}^N)} - \lambda, \|u_0\|_{L^\infty(\mathbb{T}^N)} + \lambda\right], \quad \forall n \in \{0, ..., N_T - 1\}, \forall K \in \mathcal{T}\Big\}. \tag{36}$$

*Remark* 4.1 (Support of $m_K^n$). For all $\omega \in \Omega$, $\xi \mapsto m_K^n(\xi)$ is compactly supported in the convex envelope of $v_K^n$, $\{v_L^n; L \in \mathcal{N}(K)\}$, $v_K^{n+1/2}$.

**Proof of Remark 4.1** If

$$\xi > \sup\left\{v_K^n,\ \{v_L^n\ : L \in \mathcal{N}(K)\}, v_K^{n+\frac{1}{2}}\right\}$$

then

$$\begin{aligned} m_K^n(\xi) &= -\frac{1}{\Delta t_n}\left(\left(v_K^{n+\frac{1}{2}} - \xi\right)^+ - (v_K^n - \xi)^+\right) \\ &\quad - \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\int_\xi^{+\infty} a_{K\to L}(\zeta, v_K^n, v_L^n)d\zeta \\ &= 0 - 0 = 0. \end{aligned}$$

If

$$\xi < \inf\left\{v_K^n,\ \{v_L^n\ : L \in \mathcal{N}(K)\}, v_K^{n+\frac{1}{2}}\right\}$$

then $m_K^n(\xi) =$

$$\begin{aligned} & -\frac{1}{\Delta t_n}\left(\left(v_K^{n+\frac{1}{2}} - \xi\right)^+ - (v_K^n - \xi)^+\right) - \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\int_\xi^{+\infty} a_{K\to L}(\zeta, v_K^n, v_L^n)d\zeta \\ =& -\frac{1}{\Delta t_n}\left(v_K^{n+\frac{1}{2}} - v_K^n\right) - \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\int_\xi^{+\infty} a_{K\to L}(\zeta, v_K^n, v_L^n)d\zeta \\ =& \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)} A_{K\to L}(v_K^n, v_L^n) - \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\int_\xi^{+\infty} a_{K\to L}(\zeta, v_K^n, v_L^n)d\zeta \\ =& \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)} A_{K\to L}(v_K^n, v_L^n) - \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\left(A_{K\to L}(v_K^n, v_L^n) - |K|L|A(\xi).n_{K,L}\right) \\ =& \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)} |K|L|A(\xi).n_{K,L} = \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}\int_{K|L} A(\xi).n_{K,L}d\mathcal{H}^{n-1}(x) \\ =& \int_K \operatorname{div}(A(\xi))\,dx = 0. \quad \square \end{aligned}$$

**Proof of Proposition 4.1.** Let us fix $\omega \in \Omega$ and verify the consistency conditions (30) and (31):

$$
\begin{aligned}
&\int_{\mathbb{R}} \big(a_{K\to L}(\xi, v, w) - |K|L|\, a(\xi).n_{K,L}\mathbf{1}_{0>\xi}\big) d\xi \\
=&\int_{\mathbb{R}} \Bigg( |K|L|\, a(\xi).n_{K,L}\mathbf{1}_{\xi<v\wedge w} - |K|L|\, a(\xi).n_{K,L}\mathbf{1}_{0>\xi} \\
&+ \partial_2 A_{K\to L}(v,\xi)\,\mathbf{1}_{v\le\xi\le w} + \partial_1 A_{K\to L}(\xi, w)\,\mathbf{1}_{w\le\xi\le v}\Bigg) d\xi \\
=&\int_{-\infty}^{0} -|K|L|\, a(\xi).n_{K,L}\mathbf{1}_{\xi\ge v\wedge w} d\xi + \int_{0}^{+\infty} |K|L|\, a(\xi).n_{K,L}\mathbf{1}_{\xi<v\wedge w} d\xi \\
&+ \int_{\mathbb{R}} \left(\partial_2 A_{K\to L}(v,\xi)\,\mathbf{1}_{v\le\xi\le w} + \partial_1 A_{K\to L}(\xi,w)\,\mathbf{1}_{w\le\xi\le v}\right) d\xi \\
=&\left(-|K|L|A(0).n_{K,L} + |K|L|A(w\wedge v).n_{K,L}\right) \\
&+ \mathbf{1}_{v\le w}\left(A_{K\to L}(v,w) - A_{K\to L}(v,v)\right) + \mathbf{1}_{v\ge w}\left(A_{K\to L}(v,w) - A_{K\to L}(w,w)\right) \\
=&-|K|L|A(0).n_{K,L} + A_{K\to L}(v,w) = A_{K\to L}(v,w)
\end{aligned}
$$

and

$$
\begin{aligned}
&a_{K\to L}(\xi, v, v) \\
=&\partial_2 A_{K\to L}(v,\xi)\,\mathbf{1}_{\{v\}}(\xi) + \partial_1 A_{K\to L}(\xi, v)\,\mathbf{1}_{\{v\}}(\xi) + |K|L|a(\xi).n_{K,L}\mathbf{1}_{\xi<v} \\
=&|K|L|a(\xi).n_{K,L}\mathbf{1}_{\xi<v} \text{ a.e. in } \xi.
\end{aligned}
$$

Equation (28) is the direct weak derivative of (34).
To finish the proof of this proposition 4.1, let us write

$$
v_K^{n+\frac{1}{2}} = v_K^n - \frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)} A_{K\to L}(v_K^n, v_L^n) = H\left(v_K^n, \{v_L^n\}_{L\in\mathcal{N}(K)}\right).
$$

$H$ is a nondecreasing function in each variable $v_L^n$ because $A_{K\to L}(v_K^n, v_L^n)$ is nonincreasing for its second variable. If $\omega \in \Omega_\lambda^b$ then each $H\left(., \{v_L^n\}_{L\in\mathcal{N}(K)}\right)$ is also nondecreasing by the CFL condition (35) . Indeed, let $b > a$

$$
\begin{aligned}
&H\left(b, \{v_L^n\}_{L\in\mathcal{N}(K)}\right) - H\left(a, \{v_L^n\}_{L\in\mathcal{N}(K)}\right) \\
=&b - a + \frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)} \left(A_{K\to L}(a, v_L^n) - A_{K\to L}(b, v_L^n)\right) \\
\ge&(b-a)\left(1 - \frac{\Delta t_n}{|K|} L_A^{\|u_0\|_\infty+\lambda} \sum_{L\in\mathcal{N}(K)} |K|L\,|\right) \ge 0.
\end{aligned}
$$

To prove (29), let us write the inequality $m_K^n(\xi) \ge 0$ as

$$
\frac{1}{\Delta t_n}\left[\eta_\xi(v_K^{n+1/2}) - \eta_\xi(v_K^n)\right] + \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)} \int_\xi^{+\infty} a_{K\to L}(\zeta, v_K^n, v_L^n) d\zeta \le 0 \tag{37}
$$

with the entropies $\eta_\xi(v) := (v-\xi)^+$. The family of entropic fluxes

$$
\left\{(v,w) \in \mathbb{R}^2 \mapsto \int_\xi^{+\infty} a_{K\to L}(\zeta, v, w) d\zeta :\ \xi \in \mathbb{R}\right\}
$$

associated with the entropies $\eta_\xi$ is then defined. Note that

$$(38)\qquad \int_\xi^{+\infty} a_{K\to L}(\zeta, v_K^n, v_L^n)d\zeta = \int_{\mathbb{R}} \eta^{'}(\zeta)a_{K\to L}(\zeta, v_K^n, v_L^n)d\zeta$$
$$= A_{K\to L}(v_K^n, v_L^n) - A_{K\to L}(v_K^n \wedge \xi, v_L^n \wedge \xi).$$

Using the fact that $\eta_\xi(v) = v - v \wedge \xi$ , we can see that (37) is equivalent to

$$-v_K^{n+\frac{1}{2}} \wedge \xi + v_K^n \wedge \xi - \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} A_{K\to L}\left(v_K^n \wedge \xi, v_L^n \wedge \xi\right) \leq 0.$$

The fact that $H$ is nondecreasing for each of its variables implies that

$$v_K^{n+\frac{1}{2}} = H\left(v_K^n, \{v_L^n\}_{L\in\mathcal{N}(K)}\right) \geq H\left(v_K^n \wedge \xi, \{v_L^n \wedge \xi\}_{\mathcal{L}\in N(K)}\right)$$

and

$$\xi = \xi - \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} A_{K\to L}\left(\xi, \xi\right) = H\left(\xi, (\xi)_{L\in\mathcal{N}(K)}\right) \geq H\left(v_K^n \wedge \xi, \{v_L^n \wedge \xi\}_{L\in\mathcal{N}(K)}\right)$$

thus

$$v_K^{n+\frac{1}{2}} \wedge \xi \geq H\left(v_K^n \wedge \xi, \{v_L^n \wedge \xi\}_{L\in\mathcal{N}(K)}\right),$$

that is (37). □

*Remark* 4.2. We need the following exponential martingale inequality to define a subspace of $\Omega_\lambda^b$ whose probability tends towards 1 when $\lambda$ tends towards $+\infty$ in order to prove the convergence in probability of the Finite Volume Method.

**Theorem 4.2.** *Assume assumption 1.2, $T \in \mathbb{R}_+^*$, then $\forall\lambda \in \mathbb{R}_+^*$:*

$$(39)\qquad \mathbb{P}\left(\sup_{t\in[0,T]} \left|\int_0^t \Phi dW(s)\right| \geq \lambda\right) \leq 2\exp\left\{-\frac{\lambda^2}{2\|\Phi\|^2_{L_2(H,\mathbb{R})}T}\right\}$$

*and for all $\lambda \in \mathbb{R}_+^*$:*

$$(40)\qquad \tau_\lambda :\ \omega \in \Omega \mapsto\ \inf\left\{t \in [0,T] : \left|\int_0^t \Phi dW(s)\right| \geq \lambda\right\}$$

*is a predictable stopping time (with the convention* $\inf(\emptyset) = T$*)*

*Proof.* The first part of the Theorem is in fact the Bernstein inequality given By Revuz and Yor (cf [36] page 153, Exercise 3.16). To obtain the second part of the theorem, it suffices to notice that

$$t \mapsto \left|\int_0^t \Phi dW(s)\right|$$

is a progressively measurable process (continuous and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$) and to apply the Debut theorem which is for example found in [1]. Such a hitting time is also predictable. □

**Definition 4.1.** Let us define $\forall\lambda \in \mathbb{R}_+^*$,

$$(41)\qquad \Omega_\lambda := \left\{\omega \in \Omega : \sup_{t\in[0,T]} \left|\int_0^t \Phi dW(s)\right| < \lambda\right\}.$$

and $\forall n \in \{0, ..., N_T\}, \forall\lambda \in \mathbb{R}_+^*$,

$$(42)\qquad \Omega_\lambda^n := \left\{\omega \in \Omega : \max_{0\leq j\leq n} \left|\int_0^{t_j} \Phi dW(s)\right| < \lambda\right\}.$$

Then by (39), we have for $n, m \in \{0, ..., N_T\}$ such that $n < m$:

$$\mathbb{P}\left(\Omega_\lambda^n\right) \geq \mathbb{P}\left(\Omega_\lambda^m\right) \geq \mathbb{P}\left(\Omega_\lambda\right) \geq 1 - 2\exp\left(-\frac{\lambda^2}{2\left\|\Phi\right\|_{L_2(H,\mathbb{R})} T}\right).$$

**Proposition 4.3.** *Let $\Omega_\lambda$ be defined by (41), $\Omega_\lambda^{N_T}$ be defined by (42), $\Omega_\lambda^b$ be defined by (36) for a fixed $\lambda \in \mathbb{R}_+^*$. We have the following inclusions:*

$$\Omega_\lambda \subset \Omega_\lambda^{N_T} \subset \Omega_\lambda^b.$$

*Proof.* Let us prove that $\forall \lambda \in \mathbb{R}_+^*$,

$$\Omega_\lambda^{N_T} \subset \Omega_\lambda^b.$$

Let us fix $\omega \in \max_{1 \leq n \leq N_T} \left| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi dW(s) \right| < \lambda$, then

$$\forall K \in \mathcal{T}_\#, \quad v_K^0(\omega) \in \big[-\|u_0\|_\infty, \|u_0\|_\infty\big].$$

It implies that

$$\forall K \in \mathcal{T}_\#, \quad v_K^{1/2}(\omega) \in \big[-\|u_0\|_\infty, \|u_0\|_\infty\big]$$

and then that

$$\forall K \in \mathcal{T}_\#, \quad v_K^1(\omega) \in \left[-\|u_0\|_\infty + \int_0^{t_1} \Phi dW(s), \|u_0\|_\infty + \int_0^{t_1} \Phi dW(s)\right].$$

If we suppose that for a fixed $n \in \{1, ..., N_T - 1\}$, we have $\forall j \in \{0, ..., n-1\}, \forall K \in \mathcal{T}_\#$,

$$v_K^j(\omega) \in \left[-\|u_0\|_\infty + \int_0^{t_{j+1}} \Phi dW(s), \|u_0\|_\infty + \int_0^{t_{j+1}} \Phi dW(s)\right],$$

we obtain that

$$\forall K \in \mathcal{T}_\#, v_K^{n-1/2}(\omega) \in \left[-\|u_0\|_\infty + \int_0^{t_n} \Phi dW(s), \|u_0\|_\infty + \int_0^{t_n} \Phi dW(s)\right]$$

and then

$$\forall K \in \mathcal{T}_\#, v_K^n(\omega) \in \left[-\|u_0\|_\infty + \int_0^{t_{n+1}} \Phi dW(s), \|u_0\|_\infty + \int_0^{t_{n+1}} \Phi dW(s)\right].$$

The induction is established. It means that

$$\forall n \in \{0, ..., N_T\}, \forall K \in \mathcal{T}_\#, v_K^n(\omega) \in \left[-\|u_0\|_\infty - \lambda, \|u_0\|_\infty + \lambda\right],$$

that is $\omega \in \Omega_\lambda^b$. □

*Remark* 4.3 (CFL condition in practice). Once the $\lambda \in \mathbb{R}_+^*$ is fixed, then the constant $L_A^{\|u_0\|_\infty + \lambda}$ is fixed for the whole Finite Volume Method, that is for all $\omega \in \Omega$, even if we prove the convergence in probability only on the subset $\Omega_\lambda$ of $\Omega_\lambda^b$.

## 5. Definition and properties of the approximate solution and the approximate generalized solution up to a stopping time

The Finite Volume Method will give an approximate solution to the solution of (1) which is piecewise constant for each $\omega \in \Omega$. The choice of the sequence of approximate generalized solutions of (1) up to a suitable stopping time is very crucial. It has to follow the assumptions of definition 2.8, in order to use the convergence theorem 2.3. It is the link approaching the solution of (1) and the approximate solution given by the Finite Volume Method. After having detailed the properties of the mesh, we define both in the next section, the approximate solutions of (1) given by the Finite Volume Method and the approximate generalized solutions of (1) up to a stopping time. The associated Young measures and random measures are also explicitly formulated.

### 5.1. Notations and definitions.

**5.1.1. The mesh details.** For a fixed final time $T > 0$, let us denote $\mathfrak{d}_T$ the set of admissible space-step and time-step, defined as follows: if $h > 0$ and $(\Delta t) = (\Delta t_0, \ldots, \Delta t_{N_T-1})$, $N_T \in \mathbb{N}^*$, then we say that $\delta := (h, (\Delta t)) \in \mathfrak{d}_T$ if

$$\frac{1}{h} \in \mathbb{N}\backslash\{0;1\}, \quad t_{N_T} := \sum_{n=0}^{N_T-1} \Delta t_n = T, \quad h + \sup_{0 \le n < N_T} \Delta t_n \le 1. \tag{43}$$

We say that $\delta \to 0$ if

$$|\delta| := h + \sup_{0 \le n < N_T} \Delta t_n \to 0. \tag{44}$$

For a given mesh parameter $\delta = (h, (\Delta t)) \in \mathfrak{d}_T$, we assume that a mesh $\mathcal{T}$ is given with the following properties:

$$\operatorname{diam}(K) \le h, \tag{45}$$

$$\alpha_N h^N \le |K|, \tag{46}$$

$$|\partial K| \le \frac{1}{\alpha_N} h^{N-1}, \tag{47}$$

for all $K \in \mathcal{T}$, where

$$\operatorname{diam}(K) = \max_{x,y \in K} |x - y|$$

is the diameter of $K$ and $\alpha_N$ is a given positive absolute constant depending on the dimension $N$ only. Note the following consequence of (46)-(47): for all $K \in \mathcal{T}$,

$$h|\partial K| \le \frac{1}{\alpha_N^2} |K|. \tag{48}$$

**5.1.2. Definitions related to the approximate solution.** We assume assumptions 3.1 on the numerical fluxes, the Finite Volume Method (23)-(24) allows to define the approximate solution $v_\delta$ of (1) for almost every $x \in \mathbb{T}^N$, for all $t \in [0, T]$ by

$$\forall x \in K, \forall t \in [t_n, t_{n+1}[, \ v_\delta(x,t) := v_K^n, \ \text{and } \forall x \in K, \ v_\delta(x,T) := v_K^{N_T-1}. \tag{49}$$

We will also need the intermediate discrete functions defined $\forall x \in \mathbb{T}^N$ by

$$v_\delta^\flat(x, t_{n+1}) := v_K^{n+1/2}, \quad \forall x \in K, \forall K \in \mathcal{T}, \forall n \in \{0, \cdots, N_T - 1\}. \tag{50}$$

Let us now introduce another approximation of the solution: the function $v_\delta^\sharp$ defined $\forall x \in K, \forall K \in \mathcal{T}, \forall t \in [0,T)$ by

$$v_\delta^\sharp(x,t) = v_K^\sharp(t) := \sum_{n\in\{0,..N_T-1\}} \mathbf{1}_{[t_n,t_{n+1})}(t)\left(v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi \, dW(s)\right) \tag{51}$$

and $\forall x \in K, \forall K \in \mathcal{T}, t = T$ by

$$v_\delta^\sharp(x,T) = v_K^\sharp(T) := v_K^{N_T-\frac{1}{2}} + \int_{t_{N_T-1}}^{T} \Phi \, dW(s).$$

The aim is to use the theorem 2.3, and to define a sequence $(f_\delta)$ of approximate generalized solutions up to a stopping time, associated with the scheme (23), (24). The facts that the $v_K^n$ are bounded on the subset $\Omega_\lambda$ defined by (41), that the probability of $\Omega_\lambda$ tends to one with $\lambda$ towards $+\infty$, and that the approximations are considered in probability allow to define the function $f_\delta^\lambda(x,t,\xi), \forall \xi \in \mathbb{R}, x \in \mathbb{T}^N, \forall n \in \{0,...,N_T-1\}, \forall t \in [t_n,t_{n+1})$, as

$$f_\delta^\lambda(x,t,\xi) := \mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\frac{t-t_n}{\Delta t_n}\mathbf{1}_{v_\delta^\sharp(x,t)>\xi} + \frac{t_{n+1}-t}{\Delta t_n}\mathbf{1}_{v_\delta(x,t)>\xi}\right) \tag{52}$$

for the predictable stopping time $\tau_\lambda$ defined by (40), and

$$f_\delta^\lambda(x,T,\xi) := \mathbf{1}_{[0,\tau_\lambda]}(t_{N_T-1})\mathbf{1}_{v_\delta^\sharp(x,T)>\xi}$$

which makes almost surely

$$F_\delta^\lambda : t \in [0,T] \mapsto \int_{\mathbb{T}^N\times\mathbb{R}} f_\delta^\lambda(x,t,\xi)\varphi(x,\xi)\, dxd\xi$$

càdlàg for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$. In fact, at fixed $\omega \in \Omega$, it has only one point of discontuity, that is at the first $t_n$ for which $\mathbf{1}_{[0,\tau_\lambda]}(t_n) = 0$. It is very important for two reasons: the $t \mapsto F_\delta^\lambda(t\wedge\tau_\lambda)$ are almost surely continuous for all $t \in [0,T]$ and it is necessary to obtain (13) from the discrete kinetic equation (62) in section 6. We can associate $f_\delta^\lambda$ with the opposite of its weak derivative $\nu_{x,t}^{\delta,\lambda}(d\xi) := -\partial_\xi f_\delta^\lambda(x,t,\xi)$ which is equal $\forall n \in \{0,...,N_T-1\},\ \forall t \in [t_n,t_{n+1}),\ \forall x \in \mathbb{T}^N$ to

$$\nu_{x,t}^{\delta,\lambda}(d\xi) = \mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\frac{t-t_n}{\Delta t_n}\delta_{v_\delta^\sharp(x,t)}(d\xi) + \frac{t_{n+1}-t}{\Delta t_n}\delta_{v_\delta(x,t)}(d\xi)\right) \tag{53}$$

and $\forall x \in \mathbb{T}^N,\ t = T$ to

$$\nu_{x,T}^{\delta,\lambda}(d\xi) = \mathbf{1}_{[0,\tau_\lambda]}(t_{N_T-1})\delta_{v_\delta^\sharp(x,T)}(d\xi).$$

We denote $m_\delta^\lambda$ the discrete random measures defined $\forall (x,t,\xi) \in \mathbb{T}^N \times [0,T] \times \mathbb{R}$ by

$$\begin{aligned} dm_\delta^\lambda(x,t,\xi) := &\sum_{n=0}^{N_T-2}\sum_{K\in\mathcal{T}_\#} \mathbf{1}_{K\times[t_n,t_{n+1})}(x,t)\mathbf{1}_{[0,\tau_\lambda]}(t_n)m_K^n(\xi)\, dxdtd\xi \\ &+ \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{K\times[t_{N_T-1},T]}(x,t)\mathbf{1}_{[0,\tau_\lambda]}(t_{N_T-1})m_K^{N_T-1}(\xi)\, dxdtd\xi. \end{aligned} \tag{54}$$

*Remark* 5.1. The CFL condition (35) implies that the inequality

$$\mathbf{1}_{[0,\tau_\lambda]}(t_n)m_K^n(\xi) \geq 0$$

holds almost surely $\forall n \in \{0,...,N_T-1\}, \forall K \in \mathcal{T}, \forall \xi \in \mathbb{R}$.

From the next section up to the end of section 6, we will prove that the assumptions of theorem 2.3 are satisfied, that is $(f_\delta^\lambda)_\delta$ is a sequence of approximate generalized solutions of (1) up to the stopping time $\tau_\lambda$, the sequence $(\nu^{\delta,\lambda})_\delta$ is verifying the bound (14), and the sequence $(m_\delta^\lambda)_\delta$ is verifying the bounds (15) and (16).

**5.2. First properties of the sequence $(f_\delta^\lambda)_\delta$.** Let us prove now that the sequence $\left(f_\delta^\lambda\right)_\delta$ satisfies the first item of definition 2.7 and the initial condition of Theorem 2.3.

**5.2.1. The first item of definition 2.7 is satisfied by the sequence $(f_\delta^\lambda)_\delta$.** At fixed $t \in [0,T]$, $\omega \in \Omega$,

- $f_\delta^\lambda\,(x,t,\xi) \in [0,1]$,
- $x \mapsto v_\delta(x,t)$ and $x \mapsto v_\delta^\sharp(x,t)$ are constant on each $K \in \mathcal{T}$. The $K$ are open sets of $(0,1)^N$ thus $v_\delta(.,t), v_\delta^\sharp(.,t) \in L^\infty(\mathbb{T}^N)$ hence $f_\delta^\lambda(.,t,.) \in L^\infty(\mathbb{T}^N \times \mathbb{R};[0,1])$.
- At fixed $\omega \in \Omega$, and fixed $t \in [0,\tau_\lambda]$, $\nu_{x,t}^{\delta,\lambda}(d\xi) = -\partial_\xi f_\delta^\lambda(x,t,\xi)$ are Young measures which vanish at infinity by the next proposition 5.2 i.e. by the proposition which gives the tightness of associated Young measures. Thus, almost surely, $\forall t \in [0,\tau_\lambda]$, $f_\delta^\lambda(.,t,.)$ are kinetic functions, thus $\forall t \in [0,T]$, $f_\delta^\lambda(.t\wedge\tau_\lambda,.)$ are kinetic functions.
- To clarify the measurability, let us write the parameter $\omega \in (\Omega,\mathcal{F})$ often omitted: for all $R>0$,

$$\mathbf{1}_{[0,\tau_\lambda]}f_\delta^\lambda : (t,\omega) \in [0,T]\times\Omega \mapsto \mathbf{1}_{[0,\tau_\lambda(\omega)]}(t)f_\delta^\lambda(.,t,.,\omega) \in L^1\left([0,1)^N\times(-R,R);\mathbb{R}\right).$$

Due to the separability of $L^1\left([0,1)^N\times(-R,R);\mathbb{R}\right)$, to show that $\mathbf{1}_{[0,\tau_\lambda]}f_\delta^\lambda$ is a predictable process, it is sufficient to prove that $\forall g \in L^\infty([0,1)^N\times(-R,R);\mathbb{R})$, the process

$$(t,\omega)\in[0,T]\times\Omega \mapsto \int_{[0,1)^N\times(-R,R)} \mathbf{1}_{[0,\tau_\lambda(\omega)]}(t)f_\delta^\lambda(x,t,\xi,\omega)g(x,\xi)dxd\xi \in \mathbb{R}$$

is predictable. By sum of measurable functions, it is sufficient for such $g$, to prove that, $\forall K \in \mathcal{T}_\#$ the process

$$L_K \;: (t,\omega)\in[0,T]\times\Omega \mapsto \int_{K\times(-R,R)} \mathbf{1}_{[0,\tau_\lambda(\omega)]}(t)f_\delta^\lambda(x,t,\xi,\omega)g(x,\xi)dxd\xi \in \mathbb{R}$$

is predictable. The process is in fact left-continuous and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, therefore this process is predictable. Then, the process

$$(t,\omega)\mapsto L_K(t\wedge\tau_\lambda(\omega),\omega)$$

is continuous and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, therefore it is predictable. By periodic extension, the process $\mathbf{1}_{[0,\tau_\lambda]}f_\delta^\lambda$ is also predictable, then the process $f_\delta^\lambda(.,t\wedge\tau_\lambda,.)$ is predictable. The fact that

$$(t,\omega)\mapsto f_\delta^\lambda(.,t\wedge\tau_\lambda(\omega,.,\omega)\in L^1([0,T]\times\Omega;L^1(\mathbb{T}^N\times(-R,R);\mathbb{R}))$$

is a consequence of the fact that $f_\delta^\lambda$ is bounded by 1.

### 5.2.2. The initial condition is satisfied by the sequence $\left(f_\delta^\lambda\right)_\delta$.

**Proposition 5.1.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume assumptions 1.1, 1.2. Let $f_\delta^\lambda$ be defined by (52). Let $T \in \mathbb{R}_+^*, \delta \in \mathfrak{d}_T, \lambda \in \mathbb{R}_+^*$, then we have $\forall \varphi \in L^1\left(\mathbb{T}^N \times \mathbb{R}\right)$,*

$$\int_{\mathbb{T}^N \times \mathbb{R}} \left(f_\delta^\lambda(x,0,\xi) - \mathbf{1}_{u_0(x)>\xi}\right) \varphi(x,\xi)\, dx d\xi \to_{|\delta|\to 0} 0.$$

*Proof.* (see also [11] page 328) By assumption (46), we have $h^N \alpha_N \leq |K|$. Denoting $\mathcal{T}_{\#,h}$ the mesh of $(0,1)^N$ associated with the value $h = \frac{1}{n}$ for $n \in \mathbb{N}^*$,

$$\bigcup_{\mathcal{T}_{\#,h} : \frac{1}{h} \in \mathbb{N}^*} \ \bigcup_{K \in \mathcal{T}_{\#,h}} K$$

can be called (in the sense of Rudin p146, [37]) substantial family because $\forall x \in K$,

$$K \subset B(x,h) \text{ and } |B(x,h)| = h^N |B(x,1)| \leq \frac{|B(x,1)|}{\alpha_N} |K|.$$

Therefore we can apply the Lebesgue differentiation theorem for almost all $x \in K$:

$$\lim_{h \to 0} \frac{1}{|K|} \int_K u_0(y)\, dy - u_0(x) = 0.$$

More precisely (see W. Rudin p151 [37]),

$$\forall \varepsilon > 0,\ \exists \delta > 0\ :\ \Big\{ \forall K\ :\ \operatorname{diam}(K) < \delta \text{ and for almost all } x \in K,\quad \frac{1}{|K|} \int_K |u_0(y) - u_0(x)|\, dy < \varepsilon \Big\}.$$

Thus, by assumption (45), $\forall K \in \mathcal{T}_{\#,h}$, $\operatorname{diam}(K) < h$. We have for $h$ small enough

$$\sum_{K \in \mathcal{T}_{\#,h}} \int_K \left| \frac{1}{|K|} \int_K u_0(y)\, dy - u_0(x) \right| dx < \varepsilon$$

and thus, $\forall \varphi \in C_c^\infty\left(\mathbb{T}^N \times \mathbb{R}\right)$, and $h$ small enough,

$$\begin{aligned}
&\int_{\mathbb{T}^N \times \mathbb{R}} \left(f_\delta^\lambda(x,0,\xi) - \mathbf{1}_{u_0(x)>\xi}\right) \varphi(x,\xi)\, dx d\xi \\
&= \int_{\mathbb{T}^N \times \mathbb{R}} \left(\mathbf{1}_{v_\delta(x,0)>\xi} - \mathbf{1}_{u_0(x)>\xi}\right) \varphi(x,\xi)\, dx d\xi \leq \|\varphi\|_\infty \int_{\mathbb{T}^N} |v_\delta(x,0) - u_0(x)|\, dx \\
&\qquad = \|\varphi\|_\infty \sum_{K \in \mathcal{T}_{\#,h}} \int_K \left| \frac{1}{|K|} \int_K u_0(y)\, dy - u_0(x) \right| dx < \|\varphi\|_\infty \times \varepsilon.
\end{aligned}$$

By density of $C_c^\infty\left(\mathbb{T}^N \times \mathbb{R}\right)$ in $L^1\left(\mathbb{T}^N \times \mathbb{R}\right)$, we also prove that $\forall \varphi \in L^1\left(\mathbb{T}^N \times \mathbb{R}\right)$,

$$\int_{\mathbb{T}^N \times \mathbb{R}} \left(f_\delta^\lambda(x,0,\xi) - \mathbf{1}_{u_0(x)>\xi}\right) \varphi(x,\xi)\, dx d\xi \to_{|\delta|\to 0} 0. \quad \square$$

### 5.3. Tightness of the sequence of Young measures $(\nu^{\delta,\lambda}_{t\wedge\tau_\lambda})_\delta$.

**Proposition 5.2** (Tightness of $(\nu^{\delta,\lambda})$). *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume assumptions 1.1, 1.2 and* (35). *Let $\tau_\lambda$ be defined by* (40). *Let $(v_\delta(t))$ be the numerical unknown defined by* (23)-(24)-(49) *and let $\nu^{\delta,\lambda}$ be defined by* (53). *Let $p \in [1,+\infty), T \in \mathbb{R}^*_+$, then we have $\forall \delta \in \mathfrak{d}_T$, and for $\lambda \in \mathbb{R}^*_+$ big enough:*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^N}\int_{\mathbb{R}}(|\xi|^p)d\nu^{\delta,\lambda}_{x,t\wedge\tau_\lambda}(\xi)dx\right) \le 2\left|\|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda\right|^p + 2^{2p-1}\lambda^p. \tag{55}$$

*Proof.* Let us find a bound to

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^N}\int_{\mathbb{R}}(|\xi|^p)\nu^{\delta,\lambda}_{x,t\wedge\tau_\lambda}(d\xi)dx\right)\\
\le&\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^N}\int_{\mathbb{R}}(|\xi|^p)\nu^{\delta,\lambda}_{x,t}(d\xi)dx\right)\\
=&\mathbb{E}\left(\max_{n\in\{0,\dots,N_T-1\}}\sup_{t\in[t_n,t_{n+1})}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p\nu^{\delta,\lambda}_{x,t}(d\xi)\,dx\right)\\
=&\mathbb{E}\left(\max_{n\in\{0,\dots,N_T-1\}}\sup_{t\in[t_n,t_{n+1})}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{\mathbb{T}^N}\left(\frac{t-t_n}{\Delta t_n}|v^\sharp_\delta(x,t)|^p\right.\right.\\
&\left.\left.+\frac{t_{n+1}-t}{\Delta t_n}|v_\delta(x,t)|^p\right)dx\right)\\
\le&\mathbb{E}\left(\max_{n\in\{0,\dots,N_T-1\}}\sup_{t\in[t_n,t_{n+1})}\int_{\mathbb{T}^N}\left(2^{p-1}\left|\|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda\right|^p\right.\right.\\
&\left.\left.+\left|\|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda\right|^p\right)dx\right)\\
&+\mathbb{E}\left(\max_{n\in\{0,\dots,N_T-1\}}\sup_{t\in[t_n,t_{n+1})}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{\mathbb{T}^N}2^{p-1}\left|\int_{t_n}^t\Phi dW(s)\right|^p dx\right)\\
\le&2\left|\|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda\right|^p\\
&+\max\left(2^{2p-1}\lambda^p;\max_{n\in\{0,\dots,N_T-1\}}\mathbb{E}\sup_{t\in[t_n,t_{n+1})}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{\mathbb{T}^N}2^{p-1}\left|\int_{t_n}^t\Phi dW(s)\right|^p dx\right)\\
\le&2\left|\|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda\right|^p+\max\left(2^{2p-1}\lambda^p;2^{p-1}C_p\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}}\|\Phi\|^2_{L_2(H,\mathbb{R})}ds\right)^{p/2}\right]\right)\\
\le&2\left|\|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda\right|^p+\max\left(2^{2p-1}\lambda^p;2^{p-1}C_p\|\Phi\|^p_{L_2(H,\mathbb{R})}\right),
\end{aligned}$$

where $C_p$ is a universal constant of the Burkholder Davis Gundy inequality and because $\forall n \in \{0,\dots,N_T-1\}, \forall K \in \mathcal{T}_\#$:

$$\mathbf{1}_{[0,\tau_\lambda]}(t_n)\,|v^n_K| \le \|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda; \mathbf{1}_{[0,\tau_\lambda]}(t_n)\left|v^{n+1/2}_K\right| \le \|u_0\|_{L^\infty(\mathbb{T}^N)}+\lambda,$$

and for $n$ such that $t \le t_{n+1} \le \tau_\lambda$:

$$\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left|\int_{t_n}^t\Phi\,dW(s)\right| \le \mathbf{1}_{[0,\tau_\lambda]}(t_n)\left|\int_0^t\Phi\,dW(s)\right|+\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left|\int_0^{t_n}\Phi\,dW(s)\right| \le 2\lambda. \quad \square$$

*Remark* 5.2. It means that the condition (14) is satisfied by the sequence $\left(f_\delta^\lambda\right)_\delta$.

### 5.4. Tightness of the sequence of kinetic measures $(m_\delta^\lambda)_\delta$.

**Lemma 5.3.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume assumptions 1.1, 1.2 and* (35)*. Let $\tau_\lambda$ be defined by* (40)*. The kinetic measures $m_\delta^\lambda$ defined on $\mathbb{T}^N \times [0,T] \times \mathbb{R}$ by* (54) *using* (24)*,* (26)*,* (27)*,* (33)*,* (34)*, are random measures.*

*Remark* 5.3. Usually the parameter $\omega$ is omitted, but to talk about random measures, we have to write it.

*Proof.* At fixed $\omega \in \Omega$, the bounded measures $m_\delta^\lambda(dx,dt,d\xi)(\omega)$ are absolutely continuous with respect to the Lebesgue measure $dxdtd\xi$ because the $m_K^n(.)$ are continuous and compactly supported. Their non-negativity is due to the remark 5.1. Let $\psi \in C_b(\mathbb{T}^N \times [0,T] \times \mathbb{R})$,

$$\omega \in \Omega \mapsto m_\delta^\lambda(\psi)(\omega) = \sum_{K\in\mathcal{T}_\#} \sum_{n=0}^{N_T-1} \int_{\mathbb{R}} \int_K \int_{t_n}^{t_{n+1}} \psi(x,t,\xi) \mathbf{1}_{[0,\tau_\lambda(\omega)]}(t_n) m_K^n(\xi,\omega) dxdtd\xi.$$

To show that it is a random variable, it suffices to show that

$$\omega \in \Omega \mapsto \int_{\mathbb{R}} \int_K \int_{t_n}^{t_{n+1}} \psi(x,t,\xi) \mathbf{1}_{[0,\tau_\lambda(\omega)]}(t_n) m_K^n(\xi,\omega) dxdtd\xi$$

is a random variable for each $n$ and each $K$. The $\mathcal{F}$-measurability of $v_K^n, v_K^{n+\frac{1}{2}}$ is proved by induction on $n$, that implies the $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurability of $m_K^n(\xi)$. The $\mathcal{F}$-measurability of $\mathbf{1}_{[0,\tau_\lambda(\omega)]}(t_n)$ is due to the fact that $\tau_\lambda$ is a stopping time. As the $\xi \mapsto m_K^n(\xi)$ are bounded and compactly supported, the $\psi(x,t,\xi)\mathbf{1}_{[0,\tau_\lambda(\omega)]}(t_n) m_K^n(\xi)$ are integrable on $\mathbb{T}^N \times [0,T] \times \mathbb{R} \times \Omega$, the Fubini's theorem allows to conclude. □

**Lemma 5.4.** *Assume assumptions 1.1, 1.2 and* (35)*. Let $u_0 \in L^\infty(\mathbb{T}^N)$, $(v_\delta(t))$ and $(v_\delta^\flat(t))$ be the numerical unknowns defined by* (23)-(24)-(49), (50)*. Let $T \in \mathbb{R}_+^*$, then we have $\forall \delta \in \mathfrak{d}_T, \forall \lambda \in \mathbb{R}_+^*$:*

(56)
$$\sum_{n=0}^{N_T-1} \mathbb{E}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\|v_\delta(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2 - \|v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2\right)\right) \le T\,\|\Phi\|_{L_2(H,\mathbb{R})}^2,$$

$$\sum_{n=0}^{N_T-1} \mathbb{E}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\|v_\delta(t_{n+1})\|_{L^4(\mathbb{T}^N)}^4 - \|v_\delta^\flat(t_{n+1})\|_{L^4(\mathbb{T}^N)}^4\right)\right)$$
$$\le 12T\,\|\Phi\|_{L_2(H,\mathbb{R})}^2 \left((\|u_0\|_\infty + \lambda)^2 + \|\Phi\|_{L_2(H,\mathbb{R})}^2\right). \tag{57}$$

*Proof.* The numerical scheme gives the equality

$$v_K^{n+1} = v_K^{n+\frac{1}{2}} + \int_{t_n}^{t_{n+1}} \Phi\; dW(s)$$

for all $n \in \{0,\cdots,N_T-1\}$. Thus, applying Itô's formula to the function $s \in \mathbb{R} \mapsto s^2$ and to the stochastic process

$$t \mapsto v_K^\sharp(t) = v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi\; dW(s)$$

defined on $[t_n, t_{n+1})$, we obtain (passing to the limit when $t$ tends to $t_{n+1}$)

$$\left(v_K^{n+1}\right)^2 = \left(v_K^{n+1/2}\right)^2 + 2\int_{t_n}^{t_{n+1}} \left(v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi \, dW(s)\right) \Phi \, dW(t) + \int_{t_n}^{t_{n+1}} \|\Phi\|^2_{L_2(H,\mathbb{R})} dt.$$

Multiplying by $\mathbf{1}_{[0,\tau_\lambda]}(t_n)$ and taking the expectation, we obtain

$$\mathbb{E}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\left(v_K^{n+1}\right)^2 - \left(v_K^{n+1/2}\right)^2\right)\right) = \Delta t_n \|\Phi\|^2_{L_2(H,\mathbb{R})}.$$

Multiplying by $|K|$, summing over the $K \in \mathcal{T}_\#$, and summing over the $n \in \{0, \cdots, N_T - 1\}$, we obtain (56). If now, we apply Itô's formula to the function $s \in \mathbb{R} \mapsto s^4$ and to the stochastic process

$$t \mapsto v_K^\sharp(t) = v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi \, dW(s)$$

defined on $[t_n, t_{n+1})$, we obtain (passing to the limit when $t$ tends to $t_{n+1}$):

$$\left(v_K^{n+1}\right)^4 = \left(v_K^{n+1/2}\right)^4 + 4\int_{t_n}^{t_{n+1}} \left(v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi \, dW(s)\right)^3 \Phi \, dW(t) + 6\int_{t_n}^{t_{n+1}} \left(v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi \, dW(s)\right)^2 \|\Phi\|^2_{L_2(H,\mathbb{R})} dt.$$

Multiplying by $\mathbf{1}_{[0,\tau_\lambda]}(t_n)$ and taking the expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\left(v_K^{n+1}\right)^4 - \left(v_K^{n+1/2}\right)^4\right)\right) \\
=&6\int_{t_n}^{t_{n+1}} \mathbb{E}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(v_K^{n+\frac{1}{2}} + \int_{t_n}^{t} \Phi \, dW(s)\right)^2\right) \|\Phi\|^2_{L_2(H,\mathbb{R})} dt \\
\leq&12\int_{t_n}^{t_{n+1}} \left(\mathbb{E}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(v_K^{n+\frac{1}{2}}\right)^2\right) + \Delta t_n \|\Phi\|^2_{L_2(H,\mathbb{R})}\right) \|\Phi\|^2_{L_2(H,\mathbb{R})} dt \\
\leq&12\Delta t_n \|\Phi\|^2_{L_2(H,\mathbb{R})}\left(\left(\|u_0\|_\infty + \lambda\right)^2 + \Delta t_n \|\Phi\|^2_{L_2(H,\mathbb{R})}\right).
\end{aligned}$$

Multiplying by $|K|$, summing over the $K \in \mathcal{T}_\#$, and summing over the $n \in \{0, \cdots, N_T - 1\}$, we obtain (57). □

*Remark* 5.4. The following proposition gives a stronger result than the expected assumptions (15) and (16) needed to use theorem 2.3. Indeed,

$$\mathbb{E}\int_{\mathbb{T}^N \times [0,T] \times \mathbb{R}} (1 + |\xi|^2) dm_\delta^\lambda(x, t, \xi) \leq Cste$$

implies

$$\mathbb{E}\int_{\mathbb{T}^N \times [0,T] \times \mathbb{R}} 1 dm_\delta^\lambda(x, t, \xi) + R^2 \mathbb{E}\int_{\mathbb{T}^N \times [0,T] \times B_R^c} 1 dm_\delta^\lambda(x, t, \xi) \leq Cste.$$

The first left-hand term of the previous inequality gives (15), the second gives (16).

**Proposition 5.5** (Tightness of $(m_\delta^\lambda)_\delta$). *Assume assumption 1.1, 1.2 and* (35). *Let* $u_0 \in L^\infty(\mathbb{T}^N)$, $T \in \mathbb{R}_+^*, \delta \in \mathfrak{d}_T$ *and* $\lambda \in \mathbb{R}_+^*$. *Let* $m_\delta^\lambda$ *be defined by* (54) *using* (24), (26), (27), (33), (34), *then we have*

$$\begin{aligned}(58)\qquad &\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}(1+|\xi|^2)dm_\delta^\lambda(x,t,\xi)\le \|u_0\|^2_{L^\infty(\mathbb{T}^N)}+T\,\|\Phi\|^2_{L_2(H,\mathbb{R})}\\ &+\|u_0\|^4_{L^\infty(\mathbb{T}^N)}+12T\,\|\Phi\|^2_{L_2(H,\mathbb{R})}\left((\|u_0\|_\infty+\lambda)^2+\|\Phi\|^2_{L_2(H,\mathbb{R})}\right).\end{aligned}$$

*Proof.* Let $p=2$ or $p=4$. We start from (28), we multiply it by $(\xi^p)^{'}=p\xi^{p-1}$, we integrate over $\mathbb{R}$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}}(\xi^p)^{'}(\mathbf{1}_{v_K^{n+\frac12}>\xi}-\mathbf{1}_{v_K^n>\xi})d\xi+\int_{\mathbb{R}}(\xi^p)^{'}\frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)}a_{K\to L}(\xi,v_K^n,v_L^n)d\xi\\ =&\Delta t_n\int_{\mathbb{R}}(\xi^p)^{'}\partial_\xi m_K^n(\xi)d\xi,\end{aligned}$$

that is

$$\begin{aligned}&\left|v_K^{n+\frac12}\right|^p-|v_K^n|^p+\int_{\mathbb{R}}(\xi^p)^{'}\frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)}a_{K\to L}(\xi,v_K^n,v_L^n)d\xi\\ =&-\Delta t_n\int_{\mathbb{R}}p(p-1)\xi^{p-2}m_K^n(\xi)d\xi\end{aligned}$$

because $\xi\mapsto m_K^n(\xi)$ is compactly supported and its support is included in $conv\{v_K^{n+\frac12}$, $v_K^n, v_L^n : L\in\mathcal{N}(K)\}$. Thus

$$\begin{aligned}&\sum_{K\in\mathcal{T}_\#}\left(|K|\left|v_K^{n+\frac12}\right|^p-|K|\,|v_K^n|^p\right)\\ &+\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}(\xi^p)^{'}\frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)}a_{K\to L}(\xi,v_K^n,v_L^n)d\xi\\ =&\sum_{K\in\mathcal{T}_\#}-\Delta t_n|K|\int_{\mathbb{R}}p(p-1)\xi^{p-2}m_K^n(\xi)d\xi\end{aligned}$$

which can be written

$$\sum_{K\in\mathcal{T}_\#}|K|\left|v_K^{n+\frac12}\right|^p+\sum_{K\in\mathcal{T}_\#}\Delta t_n|K|\int_{\mathbb{R}}p(p-1)\xi^{p-2}m_K^n(\xi)d\xi=\sum_{K\in\mathcal{T}_\#}|K|\,|v_K^n|^p$$

because $A_{K\to L}$ being locally Lipschitz-continuous, for almost all $\xi\in\mathbb{R}$

- $\partial_2 A_{L\to K}(v_L^n,\xi)=\lim_{s\to\xi}\dfrac{A_{L\to K}(v_L^n,s)-A_{L\to K}(v_L^n,\xi)}{s-\xi}$
  $=\lim_{s\to\xi}\dfrac{-A_{K\to L}(s,v_L^n)+A_{K\to L}(\xi,v_L^n)}{s-\xi}=-\partial_1 A_{K\to L}(\xi,v_L^n)$,
- $\partial_1 A_{L\to K}(\xi,v_K^n)=-\partial_2 A_{K\to L}(v_K^n,\xi)$,

and

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_\#}|K|\frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)}a_{K\to L}(\xi,v_K^n,v_L^n)\\
=&\frac{1}{2}\sum_{K\in\mathcal{T}_\#}|K|\frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)}\left(a_{K\to L}(\xi,v_K^n,v_L^n)+a_{L\to K}(\xi,v_L^n,v_K^n)\right)\\
=&\frac{1}{2}\sum_{K\in\mathcal{T}_\#}|K|\frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)}\Big(|K|L|a(\xi).n_{K,L}\mathbf{1}_{\xi<(v_K^n\wedge v_L^n)}+\partial_2A_{K\to L}(v_K^n,\xi)\mathbf{1}_{v_K^n\le\xi\le v_L^n}\\
&+\partial_1A_{K\to L}(\xi,v_L^n)\mathbf{1}_{v_L^n\le\xi\le v_K^n}+|K|L|a(\xi).n_{L,K}\mathbf{1}_{\xi<(v_K^n\wedge v_L^n)}\\
&+\partial_2A_{L\to K}(v_L^n,\xi)\mathbf{1}_{v_L^n\le\xi\le v_K^n}+\partial_1A_{L\to K}(\xi,v_K^n)\mathbf{1}_{v_K^n\le\xi\le v_L^n}\Big)=0.
\end{aligned}$$

Finally,

$$\|v_\delta^\flat(t_{n+1})\|_{L^p(\mathbb{T}^N)}^p+\sum_{K\in\mathcal{T}_\#}\Delta t_n|K|\int_{\mathbb{R}}p(p-1)\xi^{p-2}m_K^n(\xi)d\xi=\|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p. \tag{59}$$

Multiplying by $\mathbf{1}_{[0,\tau_\lambda]}(t_n)$, summing over $n\in\{0,\dots,N_T-1\}$, we obtain

$$\begin{aligned}
&p(p-1)\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}\xi^{p-2}m_\delta^\lambda(dx,dt,d\xi,\omega)\\
&\le\sum_{n=0}^{N_T-1}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\|v_\delta(t_n)\|_{L^p(\mathbb{T}^N)}^p-\|v_\delta^\flat(t_{n+1})\|_{L^p(\mathbb{T}^N)}^p\right).
\end{aligned} \tag{60}$$

Taking the expectation, and using Lemma (5.4), we obtain for $p=2$

$$\begin{aligned}
&2\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}1m_\delta^\lambda(dx,dt,d\xi)\\
\le&\mathbb{E}\sum_{n=0}^{N_T-1}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\|v_\delta(t_n)\|_{L^2(\mathbb{T}^N)}^2-\|v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2\right)\\
\le&\mathbb{E}\sum_{n=0}^{N_T-1}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\|v_\delta(t_n)\|_{L^2(\mathbb{T}^N)}^2-\mathbf{1}_{[0,\tau_\lambda]}(t_{n+1})\|v_\delta(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2\right)\\
&+\mathbb{E}\sum_{n=0}^{N_T-1}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\|v_\delta(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\|v_\delta^\flat(t_{n+1})\|_{L^2(\mathbb{T}^N)}^2\right)\\
\le&\mathbb{E}\|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2+T\|\Phi\|_{L_2(H,\mathbb{R})}^2\le\|u_0\|_{L^\infty(\mathbb{T}^N)}^2+T\|\Phi\|_{L_2(H,\mathbb{R})}^2.
\end{aligned} \tag{61}$$

For $p=4$, we also take the expectation in (60), and using Lemma (5.4), we obtain

$$
\begin{aligned}
&12\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}\xi^2 m_\delta^\lambda(dx,dt,d\xi)\\
\le&\mathbb{E}\sum_{n=0}^{N_T-1}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\|v_\delta(t_n)\|^4_{L^4(\mathbb{T}^N)}-\|v^\flat_\delta(t_{n+1})\|^4_{L^4(\mathbb{T}^N)}\right)\\
\le&\mathbb{E}\sum_{n=0}^{N_T-1}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\|v_\delta(t_n)\|^4_{L^4(\mathbb{T}^N)}-\mathbf{1}_{[0,\tau_\lambda]}(t_{n+1})\|v_\delta(t_{n+1})\|^4_{L^4(\mathbb{T}^N)}\right)\\
&+\mathbb{E}\sum_{n=0}^{N_T-1}\left(\mathbf{1}_{[0,\tau_\lambda]}(t_n)\|v_\delta(t_{n+1})\|^4_{L^4(\mathbb{T}^N)}-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\|v^\flat_\delta(t_{n+1})\|^4_{L^4(\mathbb{T}^N)}\right)\\
\le&\mathbb{E}\|v_\delta(0)\|^4_{L^4(\mathbb{T}^N)}+12T\|\Phi\|^2_{L_2(H,\mathbb{R})}\left((\|u_0\|_\infty+\lambda)^2+\|\Phi\|^2_{L_2(H,\mathbb{R})}\right)\\
\le&\|u_0\|^4_{L^\infty(\mathbb{T}^N)}+12T\|\Phi\|^2_{L_2(H,\mathbb{R})}\left((\|u_0\|_\infty+\lambda)^2+\|\Phi\|^2_{L_2(H,\mathbb{R})}\right). \qquad\square
\end{aligned}
$$

*Remark* 5.5. It means that conditions (15) and (16) are also satisfied by the sequence of random measures $\left(\tilde{m}_\delta^\lambda\right)_\delta$ defined by

$$d\tilde{m}_\delta^\lambda(x,t,\xi):=dm_\delta^\lambda(x,t\wedge\tau_\lambda,\xi),\ \forall(x,t,\xi)\in\mathbb{T}^N\times[0,T]\times\mathbb{R},$$

where $m_\delta^\lambda$ is given by (54) using (24), (26), (27), (33), (34). Indeed, we have almost surely $d\tilde{m}_\delta^\lambda(x,t,\xi)=dm_\delta^\lambda(x,t,\xi)$.

## 6. Approximate kinetic equation

In this section, we prove that $\left(f_\delta^\lambda\right)_\delta$ defined by (52) and $\left(m_\delta^\lambda\right)_\delta$ defined by (54) verify the approximate kinetic equation (13) up to the stopping time $\tau_\lambda$ defined in (40). For that, we need the following discrete kinetic equation true on each interval $[t_n,t_{n+1})$:

**Proposition 6.1** (Discrete kinetic equation)**.** *Assume assumptions 1.1 1.2 and* (35)*. Let* $u_0\in L^\infty(\mathbb{T}^N)$, $T\in\mathbb{R}_+^*$, $\delta\in\mathfrak{d}_T$ *and* $\lambda\in\mathbb{R}_+^*$*. Let* $\tau_\lambda$, $\left(f_\delta^\lambda\right)_\delta$, $v_K^\sharp(t)$, $a_{K\to L}$, $m_K^n$ *be defined by* (40), (52), (51), (33), (34)*, respectively. Then we have* $\forall t\in[t_n,t_{n+1}), n\in\{0,...,N_T-1\}, x\in K, K\in\mathcal{T}, \psi\in C_c^\infty(\mathbb{R})$*:*

(62)

$$
\begin{aligned}
&\int_{\mathbb{R}}f_\delta^\lambda(x,t,\xi)\,\psi(\xi)d\xi dx-\int_{\mathbb{R}}f_\delta^\lambda(x,t_n,\xi)\,\psi(\xi)d\xi dx\\
=&-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{t_n}^t\left(\frac{1}{|K|}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)}a_{K\to L}\left(\xi,v_K^n,v_L^n\right)\psi(\xi)\,d\xi\right.\\
&\left.+\int_{\mathbb{R}}\partial_\xi\psi(\xi)\,m_K^n(\xi)\,d\xi\right)ds\\
&+\mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\left(\int_{t_n}^t\psi\left(v_K^\sharp(s)\right)\Phi\ dW(s)+\frac{1}{2}\|\Phi\|^2_{L_2(H,\mathbb{R})}\int_{t_n}^t\partial_\xi\psi\left(v_K^\sharp(s)\right)ds\right).
\end{aligned}
$$

*Proof.* We have for all $t \in [t_n, t_{n+1}), n \in \{0, ..., N_T - 1\}, x \in K, K \in \mathcal{T}$:

$$\begin{aligned}
&\int_{\mathbb{R}} f_\delta^\lambda(x,t,\xi)\,\psi(\xi)d\xi dx - \int_{\mathbb{R}} f_\delta^\lambda(x,t_n,\xi)\,\psi(\xi)d\xi dx \\
=&\int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\mathbf{1}_{v_K^\#(t)>\xi}\psi(\xi)\,d\xi - \int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\mathbf{1}_{v_K^n>\xi}\psi(\xi)\,d\xi \\
=&\int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\left(\mathbf{1}_{v_\delta^\#(x,t)>\xi} - \mathbf{1}_{v_K^{n+\frac12}>\xi}\right)\psi(\xi)\,d\xi \\
&+\int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\left(\mathbf{1}_{v_K^{n+\frac12}>\xi} - \mathbf{1}_{v_K^n>\xi}\right)\psi(\xi)\,d\xi \\
=&\mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\left(\int_0^{v_\delta^\#(x,t)}\psi(\xi)\,d\xi - \int_0^{v_K^{n+\frac12}}\psi(\xi)\,d\xi\right) \\
&+\int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\times(t-t_n)\left(-\frac{1}{|K|}\sum_L a_{K\to L}(\xi, v_K^n, v_L^n) + \partial_\xi m_K^n(\xi)\right)\psi(\xi)\,d\xi \\
=&\mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\left(\int_{t_n}^t \psi\left(v_K^{n+\frac12} + \int_{t_n}^s \Phi dW(r)\right)\Phi dW(s)\right. \\
&\left.+\frac12\|\Phi\|^2_{L_2(H,\mathbb{R})}\int_{t_n}^t \psi'\left(v_K^{n+\frac12} + \int_{t_n}^s \Phi dW(r)\right)ds\right) \\
&-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\int_{t_n}^t\int_{\mathbb{R}}\frac{1}{|K|}\sum_L a_{K\to L}(\xi, v_K^n, v_L^n)\,\psi(\xi)\,d\xi ds\right. \\
&\left.+\int_{t_n}^t\int_{\mathbb{R}} m_K^n(\xi)\,\psi'(\xi)\,d\xi ds\right),
\end{aligned}$$

applying Itô's formula to the process $v_K^\sharp(t)$ and the function $r \in \mathbb{R} \mapsto \int_0^r \psi(\xi)\,d\xi$. □

**6.1. Calculations leading to the approximate kinetic equation** (13) **up to the stopping time $\tau_\lambda$.** Let us apply the discrete kinetic equation (62) to any fixed $x \in K$, $\xi \mapsto \varphi(x,\xi) \in C_c^\infty(\mathbb{R})$ and $n \in \{0, ..., N_T - 1\}, t \in [t_n, t_{n+1})$ such that $t \leq \tau_\lambda$:

$$\begin{aligned}
&\int_{\mathbb{R}} f_\delta^\lambda(x,t,\xi)\,\varphi(x,\xi)\,d\xi - \int_{\mathbb{R}} f_\delta^\lambda(x,t_n,\xi)\,\varphi(x,\xi)\,d\xi \\
=&-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{t_n}^t\left(\frac{1}{|K|}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a_{K\to L}(\xi, v_K^n, v_L^n)\,\varphi(x,\xi)\,d\xi\right. \\
&\left.+\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)\,m_K^n(\xi)\,d\xi\right)ds \\
&+\mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\left(\int_{t_n}^t \varphi\left(x, v_K^\sharp(s)\right)\Phi\ dW(s)\right. \\
&\left.+\frac12\|\Phi\|^2_{L_2(H,\mathbb{R})}\int_{t_n}^t \partial_\xi\varphi\left(x, v_K^\sharp(s)\right)ds\right).
\end{aligned}$$

We integrate over $K \in \mathcal{T}_\#$, then we sum over the $K \in \mathcal{T}_\#$:

$$\begin{aligned}
&\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_\delta^\lambda(x,t,\xi)\varphi(x,\xi)\,d\xi dx - \int_{\mathbb{T}^N}\int_{\mathbb{R}} f_\delta^\lambda(x,t_n,\xi)\varphi(x,\xi)\,d\xi dx\\
=&-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\sum_{K\in\mathcal{T}_\#}\int_K \frac{1}{|K|}\int_{t_n}^t\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a_{K\to L}(\xi,v_K^n,v_L^n)\varphi(x,\xi)\,d\xi ds dx\\
&-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\sum_{K\in\mathcal{T}_\#}\int_K\int_{t_n}^t\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)\,m_K^n(\xi)\,d\xi ds dx\\
&+\mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{t-t_n}{\Delta t_n}\sum_{K\in\mathcal{T}_\#}\int_K\int_{t_n}^t \varphi\left(x,v_K^\sharp(s)\right)\Phi\;dW(s)dx\\
&+\mathbf{1}_{[0,\tau_\lambda]}(t_n)\frac{1}{2}\frac{t-t_n}{\Delta t_n}\sum_{K\in\mathcal{T}_\#}\int_K\int_{t_n}^t\partial_\xi\varphi\left(x,v_K^\sharp(s)\right)\|\Phi\|_{L_2(H,\mathbb{R})}^2\,ds dx.
\end{aligned}$$

We can invert $\int_K$ and $\int_{t_n}^t$ in the first, second, and forth term of the right-hand side of the previous equality by Fubini's theorem, and by stochastic Fubini's theorem in the third term using the fact that

$$\begin{aligned}
&\int_K \mathbb{E}\int_{t_n}^t\left\|\varphi\left(x,v_K^\sharp(s,\omega)\right)\Phi\right\|_{L_2(H,\mathbb{R})}^2 ds dx\\
\le&\,\|\varphi\|_\infty^2\int_K\mathbb{E}\int_{t_n}^t\|\Phi\|_{L_2(H,\mathbb{R})}^2\,ds dx\; < +\infty.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_\delta^\lambda(x,t,\xi)\varphi(x,\xi)\,d\xi dx - \int_{\mathbb{T}^N}\int_{\mathbb{R}} f_\delta^\lambda(x,t_n,\xi)\varphi(x,\xi)\,d\xi dx\\
=&-\mathbf{1}_{[0,\tau_\lambda]}(t_n)\sum_{K\in\mathcal{T}_\#}\int_{t_n}^t\int_K\frac{1}{|K|}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a_{K\to L}(\xi,v_K^n,v_L^n)\varphi(x,\xi)\,d\xi dx ds\\
&-\int_{t_n}^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)\,m_\delta^\lambda(dx,ds,d\xi)\\
&+\sum_{K\in\mathcal{T}_\#}\int_{t_n}^t\int_K\int_{\mathbb{R}}\varphi(x,\xi)\,\mu_{x,s,t}^\delta(d\xi)\,dx\;\Phi\;dW(s)\\
&+\frac{1}{2}\sum_{K\in\mathcal{T}_\#}\int_{t_n}^t\int_K\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)\,\|\Phi\|_{L_2(H,\mathbb{R})}^2\,\mu_{x,s,t}^\delta(d\xi)\,dx ds,
\end{aligned}$$

where $\mu_{x,s,t}^\delta$ is the Borel measure on $\mathbb{R}$ defined $\forall x\in\mathbb{T}^N, t\in[0,T], s\in[0,t)$ by
(63)

$$\mu_{x,s,t}^\delta(d\xi) := \sum_{n=0}^{N_T-1}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\mathbf{1}_{[t_n,t_n\vee(t\wedge t_{n+1}))}(s)\,\frac{t_{n+1}\wedge t - t_n\wedge t}{\Delta t_n}\delta_{v_\delta^\sharp(x,s)}(d\xi).$$

We then sum over the $n$ from 0 to the integer $l$ such that $t\in[t_l,t_{l+1})$ or $t\in[t_{N_T-1},T]$ to obtain (using the previous formula at the limit when $t \underset{<}{\to} t_{n+1}$, for

$n < l$ at $t \in [t_l, t_{l+1})$ or $t \in [t_{N_T-1}, T]$):

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} f_\delta^\lambda (x,t,\xi) \varphi(x,\xi) \, d\xi dx - \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_\delta^\lambda (x,0,\xi) \varphi(x,\xi) \, d\xi dx$$
$$= - \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \sum_{K \in \mathcal{T}_\#} \int_{t\wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K\to L}(\xi, v_K^n, v_L^n) \frac{1}{|K|} \int_K \varphi(x,\xi) \, dx d\xi ds$$
$$- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \, m_\delta^\lambda (dx, ds, d\xi)$$
$$+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \varphi(x,\xi) \, \mu_{x,s,t}^\delta (d\xi) \, dx \; \Phi \; dW(s)$$
$$+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \, \|\Phi\|^2_{L_2(H,\mathbb{R})} \, \mu_{x,s,t}^\delta (d\xi) \, dx ds.$$

This equation is true for all $t \in [0, \tau_\lambda]$. It remains to compare the first, the third and the forth terms of the righthand side of this equality to the three corresponding terms of the approximate kinetic equation (13) up to the stopping time $\tau_\lambda$ for $t \in [0, \tau_\lambda]$ and to show that their differences have the property (12). The term

$$\int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \, m_\delta^\lambda (dx, ds, d\xi)$$

is already equal for all $t \in [0, \tau_\lambda]$ to $m_\delta^\lambda(\partial_\xi \varphi)([0, t \wedge \tau_\lambda])$.

**6.2. Comparison of the fluxes.** For the readability of the sequel of the paper, we will denote $L_A^\lambda$ the constant $L_A^{\|u_0\|_\infty + \lambda}$ used in (20) and (35).

**Proposition 6.2** (space consistency)**.** *Assume assumptions 1.1, 1.2 and* (35)*. Let* $u_0 \in L^\infty(\mathbb{T}^N)$, $T \in \mathbb{R}_+^*, \delta \in \mathfrak{d}_T$ *and* $\lambda \in \mathbb{R}_+^*$*. Let* $\tau_\lambda, \left(f_\delta^\lambda\right)_\delta, a_{K\to L}$ *be defined by* (40), (52), (33)*, respectively. For all* $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$ *supported in* $\mathbb{T}^N \times \Lambda$ *with* $\Lambda$ *compact of* $\mathbb{R}$*, we have*

(64)
$$\int_{\mathbb{T}^N} \int_0^t \int_{\mathbb{R}} f_\delta^\lambda(x,s,\xi) \times a(\xi) \cdot \nabla_x \varphi(x,\xi) dx ds d\xi$$
$$= - \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \sum_{K \in \mathcal{T}_\#} \int_{t\wedge t_n}^{t \wedge t_{n+1}} \int_{\mathbb{R}} \sum_{L \in \mathcal{N}(K)} a_{K\to L}(\xi, v_K^n, v_L^n) \frac{1}{|K|} \int_K \varphi(x,\xi) \, dx d\xi ds$$
$$+ \varepsilon^\delta_{\text{space},0}(t,\varphi) + \varepsilon^\delta_{\text{space},1}(t,\varphi),$$

*almost surely, for all* $t \in [0, \tau_\lambda]$*, with the estimates*

$$\mathbb{E}\left( \sup_{t \in [0,\tau_\lambda]} \left| \varepsilon^\delta_{space,0}(t,\varphi) \right| \right) \tag{65}$$
$$\leq \sup_{\xi \in \Lambda} |a(\xi)| \times \||\nabla \varphi|\|_\infty \Bigg( T \|\Phi\|_{L_2(H,\mathbb{R})} \sqrt{\sup_{0 \leq n < N_T} \Delta t_n}$$
$$+ \sup_{n \in \{0,\dots,N_T-1\}} \Delta t_n T \Big( L_A^\lambda + \sup_{\xi \in [-\|u_0\|_\infty - \lambda, \|u_0\|_\infty + \lambda]} |A(\xi)| \Big) \Bigg),$$

*and*

$$\mathbb{E}\left( \sup_{t \in [0,\tau_\lambda]} |\varepsilon^\delta_{\text{space},1}(t,\varphi)| \right) \leq h \sup_{x \in \mathbb{T}^N, \xi \in \mathbb{R}} |\nabla_x \varphi(x,\xi)| \, 4 L_A^\lambda T \left( \|u_0\|_\infty + \lambda \right). \tag{66}$$

*Proof.* The aim is to show that for $t \in [0, \tau_\lambda]$,

$$-\sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \sum_{K\in\mathcal{T}_\#} \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{L\in\mathcal{N}(K)} a_{K\to L}\left(\xi, v_K^n, v_L^n\right) \frac{1}{|K|} \int_K \varphi\left(x,\xi\right) dx d\xi ds$$

is close to

$$\int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_\delta^\lambda\left(x,s,\xi\right) \times a\left(\xi\right).\nabla_x \varphi\left(x,\xi\right) d\xi dx ds.$$

Let us start by showing that

$$\sum_{n=0}^{N_T-1} \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \mathbf{1}_{v_\delta(x,s)>\xi} \times a\left(\xi\right).\nabla_x \varphi\left(x,\xi\right) d\xi dx ds$$

is close to

$$-\sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \sum_{K\in\mathcal{T}_\#} \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{L\in\mathcal{N}(K)} a_{K\to L}\left(\xi, v_K^n, v_L^n\right) \frac{1}{|K|} \int_K \varphi\left(x,\xi\right) dx d\xi ds.$$

For that, we use the following equalities:

$$\begin{aligned}
&\sum_{n=0}^{N_T-1} \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \mathbf{1}_{v_\delta(x,s)>\xi} \times a\left(\xi\right).\nabla_x \varphi\left(x,\xi\right) d\xi dx ds \\
&= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi} \times a\left(\xi\right). \int_K \nabla_x \varphi\left(x,\xi\right) dx d\xi ds \\
&= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi} \times a\left(\xi\right). \int_{\partial K} \varphi\left(x,\xi\right) n_K d\mathcal{H}^{N-1}\left(x\right) d\xi ds \\
&= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi} \\
&\quad \times \sum_{L\in\mathcal{N}(K)} |K|L| a\left(\xi\right). \frac{1}{|K|L|} \int_{K|L} \varphi\left(x,\xi\right) n_{K,L} d\mathcal{H}^{N-1}\left(x\right) d\xi ds \\
&= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi} \times \sum_{L\in\mathcal{N}(K)} |K|L| a(\xi).\varphi_{K|L}(\xi) n_{K,L} d\xi ds,
\end{aligned}$$

where $\varphi_{K|L}(\xi)$ stands for

$$\frac{1}{|K|L|} \int_{K|L} \varphi\left(x,\xi\right) d\mathcal{H}^{N-1}\left(x\right).$$

Then we notice that

$$\begin{aligned}
&\sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \sum_{K\in\mathcal{T}_\#} \int_{\mathbb{R}} \sum_{L\in\mathcal{N}(K)} a_{K\to L}\left(\xi, v_K^n, v_L^n\right) \varphi_{K|L}\left(\xi\right) d\xi ds \\
=&\frac{1}{2} \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \sum_{K\in\mathcal{T}_\#} \int_{\mathbb{R}} \sum_{L\in\mathcal{N}(K)} \Bigg( a_{K\to L}\Big(\xi, v_K^n, v_L^n\Big) \varphi_{K|L}\left(\xi\right) \\
&+ a_{L\to K}\left(\xi, v_L^n, v_K^n\right) \varphi_{L|K}\left(\xi\right) \Bigg) d\xi ds = 0
\end{aligned}$$

and that

$$\sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi} \times \sum_{L\in\mathcal{N}(K)} |K|L|a(\xi).\varphi_K(\xi)\, n_{K,L} d\xi ds$$

$$= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi}\ \varphi_K(\xi)$$

$$\times \sum_{L\in\mathcal{N}(K)} \int_{K|L} a(\xi) .n_{K,L} d\mathcal{H}^{N-1}(y)\, d\xi ds$$

$$= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi}\ \varphi_K(\xi) \times \int_{\partial K} a(\xi) .n_K d\mathcal{H}^{N-1}(y)\, d\xi ds$$

$$= \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi}\ \varphi_K(\xi) \times \int_{K} \operatorname{div}_y(a(\xi))\, dy d\xi ds = 0,$$

where $\varphi_K(\xi)$ stands for

$$\frac{1}{|K|}\int_K \varphi(x,\xi)\, d\mathcal{H}^{N-1}(x),$$

to compare

$$\sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{K\in\mathcal{T}_\#} \mathbf{1}_{v_K^n>\xi}$$

$$\times \sum_{L\in\mathcal{N}(K)} |K|L|a(\xi).\left(\varphi_{K|L}(\xi) - \varphi_K(\xi)\right) n_{K,L} d\xi ds$$

and

$$\sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \sum_{K\in\mathcal{T}_\#} \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{R}} \sum_{L\in\mathcal{N}(K)} a_{K\to L}(\xi, v_K^n, v_L^n) \left(\varphi_{K|L}(\xi) - \varphi_K(\xi)\right) d\xi ds.$$

Now, let us compare $a_{K\to L}(\xi, v_K^n, v_L^n)$ and $\mathbf{1}_{v_K^n>\xi}|K|L|a(\xi).n_{K,L}$ for $n$ such that $t_n \leq \tau_\lambda$. $\varphi \in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$ implies that $\xi \mapsto \varphi_{K|L}(\xi)$ and $\xi \mapsto \varphi_K(\xi)$ are also $C_c^\infty(\mathbb{R})$. So let us continue with $\gamma \in C_c^\infty(\mathbb{R})$ in place of $\varphi_{K|L}$ or $\varphi_K$.
If $v_K^n \leq v_L^n$,

$$\int_{\mathbb{R}} \gamma(\xi)\left(\mathbf{1}_{v_K^n>\xi}|K|L|a(\xi).n_{K,L} - a_{K\to L}(\xi, v_K^n, v_L^n)\right) d\xi$$

$$= \int_{\mathbb{R}} \gamma(\xi)\left(-\partial_2 A_{K\to L}(v_K^n,\xi)\,\mathbf{1}_{v_K^n\leq\xi\leq v_L^n}\right) d\xi$$

$$\leq \int_{\mathbb{R}} |\gamma(\xi)| \times |K|L|\, L_A^\lambda \mathbf{1}_{v_K^n\leq\xi\leq v_L^n}\ d\xi$$

because

$$a_{K\to L}(\xi, v_K^n, v_L^n) = |K|L|a(\xi).n_{K,L}\mathbf{1}_{v_K^n\wedge v_L^n>\xi} + \partial_2 A_{K\to L}(v_K^n,\xi)\,\mathbf{1}_{v_K^n\leq\xi\leq v_L^n} + \partial_1 A_{K\to L}(\xi, v_L^n)\,\mathbf{1}_{v_L^n\leq\xi\leq v_K^n}.$$

If $v_K^n \geq v_L^n$

$$\begin{aligned}
&\int_{\mathbb{R}} \gamma(\xi)\left(|K|L| a(\xi).n_{K,L} \mathbf{1}_{v_K^n>\xi} - a_{K\to L}\left(\xi, v_K^n, v_L^n\right)\right) d\xi \\
=&\int_{\mathbb{R}} \gamma(\xi)\left(|K|L| a(\xi).n_{K,L}\left(\mathbf{1}_{v_K^n>\xi} - \mathbf{1}_{v_L^n>\xi}\right) - \partial_1 A_{K\to L}\left(\xi, v_L^n\right)\mathbf{1}_{v_L^n\leq\xi\leq v_K^n}\right) d\xi \\
\leq&\int_{\mathbb{R}} |\gamma(\xi)| \times \mathbf{1}_{v_L^n\leq\xi\leq v_K^n} |K|L| \, 2L_A^\lambda d\xi.
\end{aligned}$$

Moreover, $\forall \xi \in \mathbb{R}$, we have

$$\begin{aligned}
\left|\varphi_{K|L}(\xi) - \varphi_K(\xi)\right| =& \left|\frac{1}{|K|L|}\int_{K|L} \varphi(x,\xi)\, d\mathcal{H}^{N-1}(x) - \frac{1}{|K|}\int_K \varphi(y,\xi)\, dy\right| \\
=&\left|\frac{1}{|K|}\int_K \frac{1}{|K|L|}\int_{K|L} \varphi(x,\xi)\, d\mathcal{H}^{N-1}(x)\, dy\right. \\
&\left. - \frac{1}{|K|L|}\int_{K|L} \frac{1}{|K|}\int_K \varphi(y,\xi)\, dy d\mathcal{H}^{N-1}(x)\right| \\
\leq&\frac{1}{|K|}\int_K \frac{1}{|K|L|}\int_{K|L} |\varphi(x,\xi) - \varphi(y,\xi)|\, d\mathcal{H}^{N-1}(x)\, dy \\
\leq& \sup_{x\in\mathbb{T}^N,\xi\in\mathbb{R}} |\nabla_x\varphi(x,\xi)|\, h.
\end{aligned}$$

Thus we find

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,\tau_\lambda]}\left|\sum_{n=0}^{N_T-1}\int_{t_n\wedge t}^{t_{n+1}\wedge t}\int_{\mathbb{T}^N} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{\mathbb{R}} \mathbf{1}_{v_\delta(x,s)>\xi} \times a(\xi).\nabla_x\varphi(x,\xi)\, d\xi dx ds\right.\right. \\
&\left.\left. - \left(-\sum_{n=0}^{N_T-1}\sum_{K\in\mathcal{T}_\#}\int_{t_n\wedge t}^{t_{n+1}\wedge t} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a_{K\to L}\left(\xi, v_K^n, v_L^n\right)\varphi_K(\xi) d\xi ds\right)\right|\right) \\
\leq& h \sup_{x\in\mathbb{T}^N,\xi\in\mathbb{R}} |\nabla_x\varphi(x,\xi)|\, 4L_A^\lambda T\left(\|u_0\|_\infty + \lambda\right).
\end{aligned}$$

Then, to show that

$$\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_\delta^\lambda(x,s,\xi) \times a(\xi).\nabla_x\varphi(x,\xi)\, d\xi dx ds$$

is close to

$$\sum_{n=0}^{N_T-1}\int_{t_n\wedge t}^{t_{n+1}\wedge t}\int_{\mathbb{T}^N}\int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\mathbf{1}_{v_\delta(x,s)>\xi} \times a(\xi).\nabla_x\varphi(x,\xi)\, d\xi dx ds,$$

let us compute for $t \in [0, \tau_\lambda]$ the two terms of this difference:

$$\begin{aligned}
&\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_\delta^\lambda(x,s,\xi) \times a(\xi).\nabla_x\varphi(x,\xi)\, d\xi dx ds \\
&- \sum_{n=0}^{N_T-1}\int_{t_n\wedge t}^{t_{n+1}\wedge t}\int_{\mathbb{T}^N}\int_{\mathbb{R}} \mathbf{1}_{[0,\tau_\lambda]}(t_n)\mathbf{1}_{v_\delta(x,s)>\xi} \times a(\xi).\nabla_x\varphi(x,\xi)\, d\xi dx ds \\
=& \sum_{n=0}^{N_T-1}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{t_n\wedge t}^{t_{n+1}\wedge t}\int_{\mathbb{T}^N}\int_{\mathbb{R}} \frac{s-t_n}{\Delta t_n}\left(\mathbf{1}_{v_\delta^\sharp(x,s)>\xi} - \mathbf{1}_{v_\delta(x,s)>\xi}\right) \\
&\times a(\xi).\nabla_x\varphi(x,\xi)\, d\xi dx ds
\end{aligned} \tag{67}$$

which gives

$$\mathbb{E}\Bigg( \sup_{t\in[0,\tau_\lambda]} \Bigg| \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \frac{s-t_n}{\Delta t_n} \left( \mathbf{1}_{v_\delta^\sharp(x,s)>\xi} - \mathbf{1}_{v_\delta(x,s)>\xi} \right)$$

$$\times\, a(\xi) . \nabla_x \varphi(x,\xi)\, d\xi dx ds \Bigg| \Bigg)$$

$$\leq \sup_{\xi\in\Lambda} |a(\xi)| \times \||\nabla\varphi|\|_\infty \times \mathbb{E}\Bigg( \sum_{n=0}^{N_T-1} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n\wedge t}^{t_{n+1}\wedge t} \int_{\mathbb{T}^N} \Big( \left| v_\delta^\sharp(x,s) - v_\delta^\flat(x,t_{n+1}) \right|$$

$$+ \left| v_\delta^\flat(x,t_{n+1}) - v_\delta(x,s) \right| \Big) dx ds \Bigg)$$

$$\leq \sup_{\xi\in\Lambda} |a(\xi)| \times \||\nabla\varphi|\|_\infty \Bigg( \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \left( \mathbb{E} \left| \int_{t_n}^{s} \Phi dW(r) \right|^2 \right)^{1/2} ds$$

$$+ \left( \sup_{n\in\{0,\dots,N_T-1\}} \Delta t_n \right) \sum_{n=0}^{N_T-1} \sum_{K\in\mathcal{T}_\#} |K| \mathbb{E}\left( \mathbf{1}_{[0,\tau_\lambda]}(t_n) \left| v_K^{n+1/2} - v_K^n \right| \right) \Bigg).$$

With

$$\sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \left( \mathbb{E} \left| \int_{t_n}^{s} \Phi dW(r) \right|^2 \right)^{1/2} ds \leq T \|\Phi\|_{L_2(H,\mathbb{R})} \sqrt{\sup_{n\in\{0,\dots,N_T-1\}} \Delta t_n},$$

it remains to give a bound to

(68)

$$\sum_{n=0}^{N_T-1} \sum_{K\in\mathcal{T}_\#} |K| \mathbb{E}\left( \mathbf{1}_{[0,\tau_\lambda]}(t_n) \left| v_K^{n+1/2} - v_K^n \right| \right)$$

$$\leq \sum_{n=0}^{N_T-1} \sum_{K\in\mathcal{T}_\#} |K| \mathbb{E}\Bigg( \mathbf{1}_{[0,\tau_\lambda]}(t_n) \frac{\Delta t_n}{|K|} \Bigg| \sum_{L\in\mathcal{N}(K)} A_{K\to L}(v_K^n, v_L^n)$$

$$- A_{K\to L}(v_K^n, v_K^n) + |K|L|\, A(v_K^n).n_{K,L} \Bigg| \Bigg)$$

$$\leq \sum_{n=0}^{N_T-1} \sum_{K\in\mathcal{T}_\#} |K| \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} \Big( |K|L|\, L_A^\lambda + |K|L| \sup_{\xi\in[-\|u_0\|_\infty-\lambda, \|u_0\|_\infty+\lambda]} |A(\xi)| \Big) \Big)$$

$$\leq T \Big( L_A^\lambda + \sup_{\xi\in[-\|u_0\|_\infty-\lambda, \|u_0\|_\infty+\lambda]} |A(\xi)| \Big) \qquad \square$$

### 6.3. Comparison of stochastic integrals.

**Proposition 6.3.** *Assume assumptions 1.1, 1.2, (35). Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T \in \mathbb{R}_+^*$, $\delta \in \mathfrak{d}_T$ and $\lambda \in \mathbb{R}_+^*$. Let $\tau_\lambda, \left(f_\delta^\lambda\right)_\delta, a_{K\to L}, v_\delta^\sharp, \nu_{x,t}^{\delta,\lambda}$ be defined by (40), (52), (33), (51), (53) respectively. Let $\mu^\delta$ be the Borel measure defined by (63). Then $\forall \varphi \in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$*

$$\int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \varphi(x,\xi)\, \mu_{x,s,t}^\delta(d\xi)\, dx\ \Phi\ dW(s)$$

$$= \int_0^t \mathbf{1}_{[0,\tau_\lambda]}(s) \int_{\mathbb{T}^N} \int_{\mathbb{R}} \varphi(x,\xi)\, \nu_{x,s}^{\delta,\lambda}(d\xi)\, dx\ \Phi\ dW(s) + \varepsilon_{\mathrm{W},2}^\delta(t,\varphi), \tag{69}$$

*almost surely, for all* $t \in [0, \tau_\lambda]$, *with the following estimation:*

$$\mathbb{E}\left(\sup_{t\in[0,\tau_\lambda]} |\varepsilon^{\delta}_{W,2}(t,\varphi)|^2\right) \leq 8\|\Phi\|^4_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T \sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n + 8\|\Phi\|^2_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T \sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n \frac{1}{\alpha_N}h^{N-1} \times \left(\left(L^\lambda_A\right)^2 + \sup_{\xi\in[-\|u_0\|_\infty-\lambda,\|u_0\|_\infty+\lambda]}|A(\xi)|^2\right). \tag{70}$$

*Proof.* Let $t \in [0, \tau_\lambda]$ and

$$\varepsilon^{\delta}_{W,2}(t,\varphi) := \int_0^t \int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi(x,\xi)\, d\mu^{\delta}_{x,s,t}(\xi)\, dx\ \Phi\ dW(s) - \int_0^t \mathbf{1}_{[0,\tau_\lambda]}(s)\int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi(x,\xi)\, d\nu^{\delta,\lambda}_{x,s}(\xi)\, dx\ \Phi\ dW(s).$$

We notice that $n \in \{0, ..., N_T-1\} \mapsto \varepsilon^{\delta}_{W,2}(t_n,\varphi)$ is a $\mathcal{F}_{t_n}$-martingale. If $t \in [t_l, t_{l+1})$ or $t \in [t_l, T]$ when $l = N_T - 1$, we can do the following decomposition:

$$\varepsilon^{\delta}_{W,2}(t,\varphi) = \varepsilon^{\delta}_{W,2}(t_l,\varphi) + \int_{t_l}^t \int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi(x,\xi)\, d\mu^{\delta}_{x,s,t}(\xi)\, dx\ \Phi\ dW(s) - \int_{t_l}^t \int_{\mathbb{T}^N}\int_{\mathbb{R}} \varphi(x,\xi)\, d\nu^{\delta,\lambda}_{x,s}(\xi)\, dx\ \Phi\ dW(s).$$

Then with a maximal inequality (see [36] page 53), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{l\in\{0..N_T-1\}} |\varepsilon^{\delta}_{W,2}(t_l,\varphi)|^2\right) \leq 4\mathbb{E}\left(|\varepsilon^{\delta}_{W,2}(t_{N_T-1},\varphi)|^2\right)\\
=&4\mathbb{E}\int_0^{t_{N_T-1}} \Bigg\| \int_{\mathbb{T}^N}\Big(\int_{\mathbb{R}} \varphi(x,\xi)\,\mu^{\delta}_{(x,s,t_{N_T-1})}(d\xi)\\
&- \int_{\mathbb{R}} \varphi(x,\xi)\,\nu^{\delta}_{x,t_{N_T-1}}(d\xi)\Big)dx\ \Phi\Bigg\|^2_{L_2(H,\mathbb{R})} ds\\
=&4\mathbb{E}\sum_{n=0}^{N_T-2}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{t_n}^{t_{n+1}}\int_{\mathbb{T}^N}\Big|\int_{\mathbb{R}} \varphi(x,\xi)\,\mu^{\delta}_{(x,s,t_{N_T-1})}(d\xi)\\
&- \int_{\mathbb{R}} \varphi(x,\xi)\,\nu^{\delta,\lambda}_{x,s}(d\xi)\Big|^2 dx\ \|\Phi\|^2_{L_2(H,\mathbb{R})}\, ds\\
=&4\mathbb{E}\sum_{n=0}^{N_T-2}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{t_n}^{t_{n+1}}\int_{\mathbb{T}^N}\Bigg|\frac{t_{n+1}-s}{\Delta t_n}\varphi\left(x, v^{\#}_\delta(x,s)\right)\\
&- \frac{t_{n+1}-s}{\Delta t_n}\varphi(x, v_\delta(x,s))\Bigg|^2 dx\ \|\Phi\|^2_{L_2(H,\mathbb{R})}\, ds\\
\leq&4\|\Phi\|^2_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty\,\mathbb{E}\sum_{n=0}^{N_T-2}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\int_{t_n}^{t_{n+1}}\int_{\mathbb{T}^N}\left|v^{\#}_\delta(x,s) - v_\delta(x,s)\right|^2 dxds\\
\leq&4\|\Phi\|^2_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty \sum_{n=0}^{N_T-2}\int_{t_n}^{t_{n+1}}\sum_{K\in\mathcal{T}_{\#}}|K|\Bigg(\mathbb{E}\left|\int_{t_n}^{s}\Phi dW(r)\right|^2
\end{aligned}$$

$$+\mathbb{E}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\Big|v_K^{n+1/2}-v_K^n\Big|^2\Big)ds$$

by an independence argument. Then, using (68) and Itô's Lemma, we obatin

$$\begin{aligned}
&\mathbb{E}\left(\sup_{l\in\{0..N_T-1\}}|\varepsilon^\delta_{W,2}(t_l,\varphi)|^2\right)\leq 4\|\Phi\|^4_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T\sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\\
&\quad+4\|\Phi\|^2_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty\sum_{n=0}^{N_T-2}\int_{t_n}^{t_{n+1}}\sum_{K\in\mathcal{T}_\#}|K|\mathbb{E}\mathbf{1}_{[0,\tau_\lambda]}(t_n)\Big|v_K^{n+1/2}-v_K^n\Big|^2ds\\
\leq&4\|\Phi\|^4_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T\sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\\
&\quad+4\|\Phi\|^2_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T\sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\frac{1}{\alpha_N}h^{N-1}\\
&\quad\times\Big(\big(L_A^\lambda\big)^2+\sup_{\xi\in[-\|u_0\|_\infty-\lambda,\|u_0\|_\infty+\lambda]}|A(\xi)|^2\Big).
\end{aligned}$$

Let $l\in\{0,...,N_T-1\}$, with almost the same computations, we give a bound to

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[t_l,t_{l+1})}|\varepsilon^\delta_{W,2}(t,\varphi)-\varepsilon^\delta_{W,2}(t_l,\varphi)|^2\right)\\
=&\mathbb{E}\sup_{t\in[t_l,t_{l+1})}\bigg|\int_{t_l}^t\Big(\int_{\mathbb{T}^N}\int_{\mathbb{R}}\varphi(x,\xi)\,\mu^\delta_{x,s,t}(d\xi)\,dx\\
&\quad-\int_{\mathbb{T}^N}\int_{\mathbb{R}}\varphi(x,\xi)\,\nu^{\delta,\lambda}_{x,s}(d\xi)\,dx\Big)\ \Phi\ dW(s)\bigg|^2\\
\leq&4\mathbb{E}\bigg|\int_{t_l}^{t_{l+1}}\Big(\int_{\mathbb{T}^N}\int_{\mathbb{R}}\varphi(x,\xi)\,\mu^\delta_{x,s,t}(d\xi)\,dx\\
&\quad-\int_{\mathbb{T}^N}\int_{\mathbb{R}}\varphi(x,\xi)\,\nu^{\delta,\lambda}_{x,s}(d\xi)\,dx\Big)\ \Phi\ dW(s)\bigg|^2\\
\leq&4\mathbb{E}\mathbf{1}_{[0,\tau_\lambda]}(t_l)\int_{t_l}^{t_{l+1}}\int_{\mathbb{T}^N}\Big|\int_{\mathbb{R}}\varphi(x,\xi)\,\mu^\delta_{x,s,t}(d\xi)\\
&\quad-\int_{\mathbb{R}}\varphi(x,\xi)\,\nu^{\delta,\lambda}_{x,s}(d\xi)\Big|^2dx\ \ \|\Phi\|^2_{L_2(H,\mathbb{R})}ds\\
\leq&4\|\Phi\|^4_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T\sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\\
&\quad+4\|\Phi\|^2_{L_2(H,\mathbb{R})}\|\partial_\xi\varphi\|^2_\infty T\sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\frac{1}{\alpha_N}h^{N-1}\\
&\quad\times\Big(\big(L_A^\lambda\big)^2+\sup_{\xi\in[-\|u_0\|_\infty-\lambda,\|u_0\|_\infty+\lambda]}|A(\xi)|^2\Big).
\end{aligned}$$

□

### 6.4. Comparison of Itô terms.

**Proposition 6.4.** *Assume assumptions 1.1, 1.2 and* (35). *Let* $u_0\in L^\infty(\mathbb{T}^N)$, $T\in\mathbb{R}^*_+,\delta\in\mathfrak{d}_T$ *and* $\lambda\in\mathbb{R}^*_+$. *Let* $\tau_\lambda,v_\delta,v^\sharp_\delta,\nu^{\delta,\lambda}_{x,t}$ *be defined by* (40), (49), (51), (53) *respectively. Let* $\mu^\delta$ *be the Borel measure defined by* (63). *Then* $\forall \varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$,

$$\begin{aligned}
(71) \qquad & \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \|\Phi\|^2_{L_2(H,\mathbb{R})} \, d\mu^\delta_{x,s,t}(\xi) dx ds \\
&= \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \|\Phi\|^2_{L_2(H,\mathbb{R})} \, d\nu^{\delta,\lambda}_{x,s}(\xi) dx ds + \varepsilon^\delta_{\mathrm{W},3}(t,\varphi),
\end{aligned}$$

*almost surely, for all* $t \in [0,\tau_\lambda]$, *with the following estimation:*

$$\begin{aligned}
(72) \quad & \mathbb{E}\left[\sup_{t\in[0,\tau_\lambda]} |\varepsilon^\delta_{\mathrm{W},3}(t,\varphi)|\right] \le 2\|\Phi\|^3_{L_2(H,\mathbb{R})} \left\|\partial^2_\xi \varphi\right\|_\infty T \sup_{n\in\{0,\dots,N_T-1\}} \sqrt{\Delta t_n} \\
& + 2\|\Phi\|^2_{L_2(H,\mathbb{R})} \left\|\partial^2_\xi \varphi\right\|_\infty T \sup_{n\in\{0,\dots,N_T-1\}} \Delta t_n \Big(L^\lambda_A + \sup_{\xi\in[-\|u_0\|_\infty-\lambda, \|u_0\|_\infty+\lambda]} |A(\xi)|\Big).
\end{aligned}$$

*Proof.* Let us define for $t \in [0,\tau_\lambda]$:

$$\begin{aligned}
\varepsilon^\delta_{W,3}(t,\varphi) :=& \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \|\Phi\|^2_{L_2(H,\mathbb{R})} \mu^\delta_{x,s,t}(d\xi)\, dx ds \\
& - \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \|\Phi\|^2_{L_2(H,\mathbb{R})} \nu^{\delta,\lambda}_{x,s}(d\xi)\, dx ds \\
=& \varepsilon^\delta_{W,3}(t_l,\varphi) + \int_{t_l}^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \|\Phi\|^2_{L_2(H,\mathbb{R})} \mu^\delta_{x,s,t}(d\xi)\, dx ds \\
& - \int_{t_l}^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi) \|\Phi\|^2_{L_2(H,\mathbb{R})} \nu^{\delta,\lambda}_{x,s}(d\xi)\, dx ds
\end{aligned}$$

with $l \in \{0,\dots,N_T-1\}$ such that $t \in [t_l, t_{l+1})$ or $t \in [t_{N_T-1}, T]$ if $l = N_T-1$. With almost the same computation as in the proof of Proposition 6.3, we obtain:

$$\begin{aligned}
& \mathbb{E}\Bigg(\sup_{l\in\{0,\dots,N_T-1\}} |\varepsilon^\delta_{W,3}(t_l,\varphi)|\Bigg) \\
\le& \mathbb{E} \sum_{n=0}^{N_T-2} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n}^{t_{n+1}} \int_{\mathbb{T}^N} \Big| \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi)\, \mu^\delta_{(x,s,t_{N_T-1})}(d\xi) \\
& - \int_{\mathbb{R}} \partial_\xi \varphi(x,\xi)\, \nu^{\delta,\lambda}_{x,s}(d\xi) \Big| dx \;\; \|\Phi\|^2_{L_2(H,\mathbb{R})} ds \\
\le& \|\Phi\|^2_{L_2(H,\mathbb{R})} \left\|\partial^2_\xi \varphi\right\|_\infty \mathbb{E} \sum_{n=0}^{N_T-2} \mathbf{1}_{[0,\tau_\lambda]}(t_n) \int_{t_n}^{t_{n+1}} \int_{\mathbb{T}^N} \Big| v^\#_\delta(x,s) - v_\delta(x,s) \Big| dx ds \\
\le& \|\Phi\|^2_{L_2(H,\mathbb{R})} \left\|\partial^2_\xi \varphi\right\|_\infty \sum_{n=0}^{N_T-2} \int_{t_n}^{t_{n+1}} \sum_{K\in\mathcal{T}_\#} |K| \Bigg( \mathbb{E}\Big| \int_{t_n}^s \Phi dW(r) \Big| \\
& + \mathbb{E}\mathbf{1}_{[0,\tau_\lambda]}(t_n) \Big| v_K^{n+1/2} - v_K^n \Big| \Bigg) ds.
\end{aligned}$$

Then, using (68) and Itô's Lemma

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{l\in\{0,\dots,N_T-1\}} |\varepsilon^{\delta}_{W,3}(t_l,\varphi)|\right)\\
&\leq \|\Phi\|^3_{L_2(H,\mathbb{R})}\left\|\partial^2_\xi\varphi\right\|_\infty T \sup_{n\in\{0,\dots,N_T-1\}}\sqrt{\Delta t_n}\\
&\quad+\|\Phi\|^2_{L_2(H,\mathbb{R})}\left\|\partial^2_\xi\varphi\right\|_\infty T \sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\Big(L^\lambda_A+\sup_{\xi\in[-\|u_0\|_\infty-\lambda,\|u_0\|_\infty+\lambda]}|A(\xi)|\Big).
\end{aligned}
$$

Let $l\in\{0,...,N_T-1\}$, with almost the same computations, we give a bound to

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[t_l,t_{l+1})} |\varepsilon^{\delta}_{W,3}(t,\varphi)-\varepsilon^{\delta}_{W,3}(t_l,\varphi)|\right)\\
\leq&\mathbb{E}\mathbf{1}_{[0,\tau_\lambda]}(t_l)\int_{t_l}^{t_{l+1}}\int_{\mathbb{T}^N}\Big|\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)\,\mu^{\delta}_{x,s,t}(d\xi)\\
&\quad-\int_{\mathbb{R}}\partial_\xi\varphi(x,\xi)\,\nu^{\delta,\lambda}_{x,s}(d\xi)\Big|dx\ \ \|\Phi\|^2_{L_2(H,\mathbb{R})}\,ds\\
\leq&\|\Phi\|^3_{L_2(H,\mathbb{R})}\left\|\partial^2_\xi\varphi\right\|_\infty T \sup_{n\in\{0,\dots,N_T-1\}}\sqrt{\Delta t_n}\\
&\quad+\|\Phi\|^2_{L_2(H,\mathbb{R})}\left\|\partial^2_\xi\varphi\right\|_\infty T \sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\Big(L^\lambda_A+\sup_{\xi\in[-\|u_0\|_\infty-\lambda,\|u_0\|_\infty+\lambda]}|A(\xi)|\Big).\ \square
\end{aligned}
$$

## 7. Convergences of the approximation given by the Finite Volume Method towards the unique solution of the time-continuous equation

**Theorem 7.1.** *Assume assumptions 1.1, 1.2. Let $u_0\in L^\infty(\mathbb{T}^N)$, $T\in\mathbb{R}^*_+,\lambda\in\mathbb{R}^*_+,\delta\in\mathfrak{d}_T$ for a mesh verifying* (35), (45), (46), (47). *Let $u$ be the solution of* (1) *defined in 2.2, with initial datum $u_0$ and let $v_\delta$ be the solution of the Finite Volume Method* (23)-(24)-(49). *Then,*

$$\forall p\in[1,+\infty),\qquad \int_0^T\|v_\delta(.,t)-u(.,t)\|^p_{L^p(\mathbb{T}^N)}dt\to_{\delta\to0}0\ \textit{in probability.}$$

*Proof.* Let the stopping time $\tau_\lambda$ be defined by (40), we can now apply Theorem 2.3 to $\left(f^\lambda_\delta\right)_\delta$ defined by (52) to obtain that $\left(z^\lambda_\delta\right)_\delta$ defined for $t\in[t_n,t_{n+1}),\forall n\in\{0,...,N_T-2\}$ and for $t\in[t_{N_T-2},T]$ by

$$z^\lambda_\delta(x,t):=\int_{\mathbb{R}}\xi d\nu^\delta_{x,t}(\xi)=\mathbf{1}_{[0,\tau_\lambda]}(t_n)\left(\frac{t-t_n}{\Delta t_n}v^\sharp_\delta(x,t)+\frac{t_{n+1}-t}{\Delta t_n}v_\delta(x,t)\right),$$

converges towards the unique solution $u(x,t\wedge\tau_\lambda)$ of (1) up to the stopping time $\tau_\lambda$, when $\delta\to0$ in the following sense (see remark 2.5 and formula (19)):
at fixed $\lambda>0$, for all $p\in[1,+\infty)$

$$\mathbb{E}\int_0^T\mathbf{1}_{[0,\tau_\lambda)}(t)\int_{\mathbb{T}^N}\left|z^\lambda_\delta(x,t)-u(x,t)\right|^p dxdt\to_{\delta\to0}0. \tag{73}$$

It is straightforward to see that

$$\mathbb{E}\int_0^T\int_{\mathbb{T}^N}\left|\mathbf{1}_{[0,\tau_\lambda)}(t)\times u(x,t)-u(x,t)\right|^p dxdt\to_{\lambda\to+\infty}0. \tag{74}$$

Then,

$$
\begin{aligned}
(75)\qquad & \mathbb{E}\int_0^T \mathbf{1}_{[0,\tau_\lambda)}(t)\int_{\mathbb{T}^N}\left|z_\delta^\lambda(x,t)-v_\delta(x,t)\right|^p dxdt\\
=&\mathbb{E}\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\mathbf{1}_{[0,\tau_\lambda)}(t)\frac{t-t_n}{\Delta t_n}\int_{\mathbb{T}^N}\left|v_\delta^\sharp(x,t)-v_\delta(x,t)\right|^p dxdt\\
\le&2^{p-1}\mathbb{E}\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\mathbf{1}_{[0,\tau_\lambda]}(t)\frac{t-t_n}{\Delta t_n}\int_{\mathbb{T}^N}\left|v_\delta^\sharp(x,t)-v_\delta^\flat(x,t_{n+1})\right|^p dxdt\\
&+2^{p-1}\mathbb{E}\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\mathbf{1}_{[0,\tau_\lambda]}(t)\frac{t-t_n}{\Delta t_n}\int_{\mathbb{T}^N}\left|v_\delta^\flat(x,t_{n+1})-v_\delta(x,t)\right|^p dxdt\\
\le&2^{p-1}\mathbb{E}\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\mathbf{1}_{[0,\tau_\lambda]}(t)\int_{\mathbb{T}^N}\left|\int_{t_n}^{t}\Phi dW(s)\right|^p dxdt\\
&+2^{p-1}\mathbb{E}\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\mathbf{1}_{[0,\tau_\lambda]}(t)\sum_{K\in\mathcal{T}_\#}|K|\left|v_K^{n+1/2}-v_K^n\right|^p dt\\
\le&2^{p-1}\sum_{n=0}^{N_T-1}\Delta t_n C_p\|\Phi\|^p(\Delta t_n)^{p/2}\\
&+\sup_{n\in\{0,\dots,N_T-1\}}\Delta t_n\times 2^{p-1}T\Big(L_A^\lambda+\sup_{\xi\in[-\|u_0\|_\infty-\lambda,\|u_0\|_\infty+\lambda]}|A(\xi)|\Big),
\end{aligned}
$$

having used (68) and the Burkholder Davis Gundy inequality ($C_p$ is an universal constant of it). Now, let us use the metric $(X,Y)\in\left(L^0(\Omega,\mathcal{F},\mathbb{P})\right)^2\mapsto\mathbb{E}\min(1,[X-Y])$ equivalent to the convergence in probability (see Touzy p41, [39]):

$$
\begin{aligned}
(76)\qquad & \mathbb{E}\min\left(1,\int_0^T\int_{\mathbb{T}^N}\left|\mathbf{1}_{[0,\tau_\lambda)}(t)\times v_\delta(x,t)-v_\delta(x,t)\right|^p dxdt\right)\\
=&\mathbb{E}\min\left(1,\int_0^T\mathbf{1}_{[\tau_\lambda,T]}(t)\left\|v_\delta(.,t)\right\|_{L^p(\mathbb{T}^N)}^p dt\right)\\
\le&\mathbb{E}\min\left(1,\int_0^T\mathbf{1}_{\tau_\lambda<T}\left\|v_\delta(.,t)\right\|_{L^p(\mathbb{T}^N)}^p dt\right)\\
\le&\mathbb{E}\min\left(1,\mathbf{1}_{\Omega_\lambda^c}\int_0^T\left\|v_\delta(.,t)\right\|_{L^p(\mathbb{T}^N)}^p dt\right).
\end{aligned}
$$

Using the exponential inequality (39) and the dominated convergence theorem, we can say that $\int_0^T\mathbf{1}_{[\tau_\lambda,T]}(t)\left\|v_\delta(.,t)\right\|_{L^p(\mathbb{T}^N)}^p dt$ converges in probability towards 0 independently of $\delta$. Using (73), (74), (75), and (76), we conclude that

$$\int_0^T\left\|v_\delta(.,t)-u(.,t)\right\|_{L^p(\mathbb{T}^N)}^p dt$$

converges in probability towards 0. □

**Theorem 7.2.** *Assume assumptions 1.1, 1.2. Let* $u_0\in L^\infty(\mathbb{T}^N)$, $T\in\mathbb{R}_+^*,\lambda\in\mathbb{R}_+^*,\delta\in\mathfrak{d}_T$ *for a mesh verifying* (35), (45), (46), (47). *Let* $u$ *be the solution of* (1) *defined in 2.2, with initial datum* $u_0$ *and let* $v_\delta$ *be the solution of the Finite Volume Method* (23)-(24)-(49)*. Then,*

$$\forall p \in [1, +\infty), \quad \int_0^T \|v_\delta(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)} dt \to_{\delta \to 0} 0 \textit{ almost surely.}$$

*Proof.* For a fixed $p \in [1, +\infty)$, another way to write the result of theorem 7.1 is: for any sequence $(\delta_i)_{i \in \mathbb{N}^*} \subset \mathfrak{d}_T$ such that $\lim_{i \to +\infty} \delta_i = 0$, the sequence

$$\left( \int_0^T \|v_{\delta_i}(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)} dt \right)_{i \in \mathbb{N}^*}$$

is converging towards 0 in probability. Let us chose a sequence $(\delta_i)_{i \in \mathbb{N}^*} \subset \mathfrak{d}_T$, there exists a subsequence $(\delta_{i_k})_{k \in \mathbb{N}^*} \subset \mathfrak{d}_T$ such that $(\int_0^T \|v_{\delta_{i_k}}(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)} dt)_{k \in \mathbb{N}^*}$ is converging towards 0 almost surely. If there exists an other subsequence $(\delta_{i_m})_{m \in \mathbb{N}^*}$ of $(\delta_i)_{i \in \mathbb{N}^*} \subset \mathfrak{d}_T$ such that

$$\left( \int_0^T \|v_{\delta_{i_m}}(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)} dt \right)_{m \in \mathbb{N}^*}$$

is not converging towards 0 almost surely, then there exists a subset $A \subset \Omega$, such that $\mathbb{P}(A) > 0$, and $\varepsilon > 0$ such that for all $N \in \mathbb{N}^*$, there exists $m_N \geq N$ which verifies

$$\int_0^T \|v_{\delta_{i_{m_N}}}(\omega,.,t) - u(\omega,.,t)\|^p_{L^p(\mathbb{T}^N)} dt \geq \varepsilon, \quad \forall \omega \in A. \tag{77}$$

But the sequence

$$\left( \int_0^T \|v_{\delta_{i_{m_N}}}(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)} dt \right)_{N \in \mathbb{N}^*}$$

is converging towards 0 in probability, thus we can find a subsequence of it which is not verifying (77). It is a contradiction.
We can thus conclude that, for all subsequence $(\delta_{i_k})_{k \in \mathbb{N}^*}$ of $(\delta_i)_{i \in \mathbb{N}^*} \subset \mathfrak{d}_T$, $\left( \int_0^T \|v_{\delta_{i_k}}(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)} dt \right)_{k \in \mathbb{N}^*}$ is converging towards 0 almost surely. □

## 8. Discussions on the mode of convergence and the assumptions of theorem 7.1

**8.1. The initial datum.** The initial datum $u_0$ is assumed to belong to $L^\infty(\mathbb{T}^N)$. It is natural because the Finite Volume Method has to converge towards the unique solution of the Cauchy problem of a particular balance law. Our solution is defined in [12], that we have rewritten in definition 2.2, which follows and slightly improves the solution of [9], which also follows and generalizes the solution of the Burger's stochastic equation in [14]. All those Cauchy problems are solved with an initial datum which is essentially bounded. It was also the case for the deterministic balance law solved by Kruzkov in [25].
The Cauchy problem is solved with an initial datum in $L^p(\mathbb{T}^N)$ for a fixed $p \in [1, +\infty)$ if the non-linear flux is globally Lipschitz continuous. It is proved in [5] with $p = 2$. The Finite volume Method is converging towards this solution with the same initial datum, and with numerical fluxes which are also globally Lipschitz continuous.
The solution of the Cauchy problem in [9] and in [12] is in $L^p(\Omega \times [0, T] \times \mathbb{T}^N)$ for

all $p \in [1,+\infty)$ with an initial datum in $L^\infty(\mathbb{T}^N)$. Even not proved, we can think that it is possible to prove uniqueness and existence of a kinetic solution with an initial datum in $L^p(\mathbb{T}^N)$ for all $p \in [1,+\infty)$. Especially, because combining the results of uniqueness in [16] and existence in [7] of a strong entropy solution, their Cauchy problem is solved with an initial datum in $L^p(\Omega \times \mathbb{T}^N)$ for all $p \in [1,+\infty)$.
The spirit of the convergence of the Finite Volume Method for hyperbolic conservation laws in deterministic cases is to work with sequences $(v_K^n)_{K\in\mathcal{T},n\in\{0,...,N_T-1\}}$ (defined by (24) and (23)) which are essentially bounded. It is taught in [40] and in [15]. It is the same spirit in stochastic cases: we try to find assumptions, conditions, ways to work with sequences which are essentially bounded. Thus, take an initial datum in $L^p(\mathbb{T}^N)$ for all $p \in [1,+\infty)$ would give an additional difficulty. Indeed, due to the Lebesgue differentiation theorem, the $v_K^0 = \frac{1}{|K|}\int_K u_0(x)dx$ are behaving like $u_0$ restricted to $K$ when $|K|$ tends towards 0, thus are not essentially bounded when $|K|$ tends towards 0. Also, Hlder's inequality applied to $v_K^0$ does not give suitable bounds.
A way which would be interesting to explore is to approach $u_0$ by a sequence $(\phi_m)_{m\in\mathbb{N}^*} \subset C^\infty(\mathbb{T}^N)$ whose elements are bounded on $\mathbb{T}^N$, to use our result of convergence of the Finite Volume Method for each $\phi_m$ as initial datum, and to prove that the sequence of solutions $(v_\delta^m)_{m\in\mathbb{N}^*}$ of the Finite Volume Method with $\phi_m$ as initial datum, tends towards the solution $v_\delta^0$ of the Finite Volume Method with $u_0 \in L^p(\mathbb{T}^N), \forall p \in [1,+\infty)$, as initial datum, when $m$ tends towards $+\infty$, independently of the mesh parameter $\delta$.

**8.2. The additive noise.** We chose an additive noise instead of a multiplicative noise in the equation (1) for the convergence of the Finite Volume Method essentially because we did not succeed to find a suitable subset $\Omega_\lambda$ of $\Omega_\lambda^b$ defined by (36), whose probability tends towards one when $\lambda$ tends towards $+\infty$, on which the solution of the Finite Volume Method could remain bounded. Indeed, if $\Phi$ were depending on the solution $u(x,t)$, the equation (1) would be written

$$du(x,t) + \mathrm{div}_x(A(u(x,t)))dt = \Phi(u(x,t))dW(t), \quad x \in \mathbb{T}^N, t \in (0,T), \tag{78}$$

with $\Phi : \mathbb{R} \to L_2(H,\mathbb{R})$ verifying for a constant $D_0 \in \mathbb{R}_+^*$:

$$\|\Phi(u)\|_{L_2(H,\mathbb{R})}^2 \le D_0, \quad \forall u \in \mathbb{R}. \tag{79}$$

This assumption would generalize (2) in assumption 1.2 or (1.4) in [13].
The Finite Volume Method would be defined $\forall n \in \{0,...,N_T-1\}, K \in \mathcal{T}$ by the formula

$$|K|(v_K^{n+1} - v_K^n) + \Delta t_n \sum_{L\in\mathcal{N}(K)} A_{K\to L}(v_K^n, v_L^n) = |K|\int_{t_n}^{t_{n+1}} \Phi(v_K^n)\, dW(s), \tag{80}$$

and the initialization

$$v_K^0 = \frac{1}{|K|}\int_K u_0(x)dx, \quad \forall K \in \mathcal{T}, \tag{81}$$

for $u_0 \in L^\infty(\mathbb{T}^N)$. But we did not succeed to find conditions to keep all the $v_K^n$ in $\Omega_\lambda^b$. For a fixed $K \in \mathcal{T}$ we can define the discrete process $M_K$ by $M_K^0 = 0$ and

$$M_K \;:\; n \in \{1,...,N_T\} \mapsto \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \Phi(u_K^m)dW(s). \tag{82}$$

It is a $\mathcal{F}_{t_n}$-martingale. $M_K$ verifying the assumptions of theorem 1.2 in [33], we have for all $K \in \mathcal{T}$:

$$\mathbb{P}\left(\sup_{n\in\{1,\dots,N_T\}} |M_K^n| \geq \lambda\right) \leq \exp\left\{-\frac{\lambda}{\sup_{n\in\{0,..,N_T-1\}} \Delta t_n}\right\},$$

thus

$$\mathbb{P}\left(\sup_{n\in\{1,\dots,N_T\},K\in\mathcal{T}} |M_K^n| \geq \lambda\right)$$
$$\leq \exp\left\{-\frac{\lambda}{\sup_{n\in\{0,..,N_T-1\}} \Delta t_n} + \ln\left(card\{K \in \mathcal{T}_\#\}\right)\right\}.$$

Even if we can take meshes which verify for any $\lambda > 1$:

$$\lim_{|\delta|\to 0} -\frac{\lambda}{\sup_{n\in\{0,..,N_T-1\}} \Delta t_n} + \ln\left(card\{K \in \mathcal{T}_\#\}\right) = -\infty,$$

we can not reproduce the proof of proposition 4.3. More precisely, we can prove that

$$\forall K \in \mathcal{T}_\#, \quad v_K^1 \in I_K^1 := \left[-\|u_0\|_\infty + \int_0^{t_1} \Phi(v_K^0)dW(s), \|u_0\|_\infty + \int_0^{t_1} \Phi(v_K^0)dW(s)\right],$$

but we do not have the next step, which would be

$$\forall K \in \mathcal{T}_\#, \quad v_K^{3/2} \in \left[-\|u_0\|_\infty + \int_0^{t_1} \Phi(v_K^0)dW(s), \|u_0\|_\infty + \int_0^{t_1} \Phi(v_K^0)dW(s)\right].$$

Indeed, the intervals

$$\left[-\|u_0\|_\infty + \int_0^{t_1} \Phi(v_L^0)dW(s), \|u_0\|_\infty + \int_0^{t_1} \Phi(v_L^0)dW(s)\right]$$

may be different for each $L \in \mathcal{N}(K)$, and different from $I_K^1$.

One of the main key of the convergence of the Finite Volume Method for deterministic hyperbolic conservation laws is to find a lower bound and an upper bound to the solution of the numerical scheme. Those lower and upper bounds are constant solutions of the Finite Volume Method.
That is why, it would be interesting to explore the same philosophy for the balance law (78) driven by a multiplicative noise with a bounded multiplying factor $\Phi$ satisfying (79). We could start by a standard real Brownian motion instead of a cylindrical Wiener process and a $\Phi : \mathbb{R} \to \mathbb{R}$ verifying $-\infty < -\Phi_1 \leq \Phi(u) \leq \Phi_1 < +\infty, \forall u \in \mathbb{R}$. We know that the Finite Volume Method is converging for such $\Phi_1 \in \mathbb{R}_+$ and $-\Phi_1 \in \mathbb{R}_-$ as it is proved in our article. Then, we could try to compare the solution of the Finite Volume Method for a balance law driven by $\Phi(u)dW$ and the solutions of the Finite Volume Method for balance laws driven by $-\Phi_1 dW$ and $\Phi_1 dW$, using a stochastic order developed in [38]. It may give the convergence of the Finite Volume Method as it is done in the deterministic case.

**8.3. The convergence in $L^p(\Omega \times \mathbb{T}^N \times [0,T])$?** If we look in detail the proof of theorem 7.1, from the convergence in probability, to get the convergence in

$L^p(\Omega \times \mathbb{T}^N \times [0,T])$, we have to improve the estimation (76) without $\min(1,.)$, which is

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \int_{\mathbb{T}^N} \left|\mathbf{1}_{[0,\tau_\lambda)}(t) \times v_\delta(x,t) - v_\delta(x,t)\right|^p dxdt\right)\\
=&\mathbb{E}\left(\int_0^T \mathbf{1}_{[\tau_\lambda,T]}(t)\, \|v_\delta(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt\right)\\
\leq&\mathbb{E}\left(\int_0^T \mathbf{1}_{\tau_\lambda<T}\, \|v_\delta(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt\right)\\
\leq&\mathbb{E}\left(\mathbf{1}_{\Omega^c_\lambda} \int_0^T \|v_\delta(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt\right)\\
=&\sum_{n=0}^{N_T-1} \sum_{K\in\mathcal{T}} \Delta t_n |K| \mathbb{E}(\mathbf{1}_{\Omega^c_\lambda} |v^n_K|^p).
\end{aligned}$$

We tried two different ways, but we did not succeed due to the CFL condition. We first tried to use the disjoint union

$$\Omega^c_\lambda = (\Omega_{2\lambda} \cap \Omega^c_\lambda) \bigcup (\Omega_{3\lambda} \cap \Omega^c_{2\lambda}) \bigcup (\Omega_{4\lambda} \cap \Omega^c_{3\lambda}) \bigcup ... = \bigcup_{i=2}^{+\infty} (\Omega_{i\lambda} \cap \Omega^c_{(i-1)\lambda}),$$

and the exponential inequality (39) to have

$$\begin{aligned}
\mathbb{E}(\mathbf{1}_{\Omega^c_\lambda} |v^n_K|^p) =& \sum_{i=2}^{+\infty} \mathbb{E}(\mathbf{1}_{\Omega_{i\lambda}\cap\Omega^c_{(i-1)\lambda}} |v^n_K|^p)\\
\leq& \sum_{i=2}^{+\infty} |\|u_0\|_\infty + i\lambda|^p\, 2\exp\left\{\frac{-(i-1)\lambda^2}{2\|\Phi\|^2_{L_2(H,\mathbb{R})} T}\right\}\\
\leq& \exp\left\{\frac{-0.5\lambda^2}{2\|\Phi\|^2_{L_2(H,\mathbb{R})} T}\right\} \sum_{i=2}^{+\infty} |\|u_0\|_\infty + i\lambda|^p\, 2\exp\left\{\frac{-(i-1.5)\lambda^2}{2\|\Phi\|^2_{L_2(H,\mathbb{R})} T}\right\},
\end{aligned}$$

which would converges towards 0 independently of $\delta$. But, we can have only one CFL condition

$$\Delta t_n \frac{|\partial K|}{|K|} L_A^{\|u_0\|_\infty + \lambda} \leq 1, \quad \forall K \in \mathcal{T}, \forall n \in \{0, \cdots, N_T - 1\}, \tag{83}$$

we can not have

$$\Delta t_n \frac{|\partial K|}{|K|} L_A^{\|u_0\|_\infty + i\lambda} \leq 1, \quad \forall K \in \mathcal{T}, \forall n \in \{0, \cdots, N_T - 1\}, \forall i \in \{2,3,4,...\}. \tag{84}$$

Then we tried to use the inequality

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \|v_\delta(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt\right)\\
\leq& 2^{p-1}\mathbb{E}\left(\int_0^T \|v_\delta(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt\right) + 2^{p-1}\mathbb{E}\left(\int_0^T \|u(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt\right),
\end{aligned}$$

with $u(.,t)$ the unique solution of (1). The second term of the right-hand side of the inequality is finite. The term $\int_0^T \|v_\delta(.,t) - u(.,t)\|^p_{L^p(\mathbb{T}^N)}\, dt$ could be bounded by a random variable which is in $L^1(\Omega)$, if we work at fixed $\omega \in \Omega$. But the CFL

condition would depend on $\omega$. That is not acceptable.
The fact is, we do not know how to control the terms $v_K^n$ when they do not belong to $\Omega_\lambda^b$ defined by (36). And finally, we do not know how they behave, even if we can think that the property 3 in definition 2.2 which is: for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u(.,t)\|^p_{L^p(\mathbb{T}^N)}\right) \leq C_p,$$

should be transferred from the solution $u(.,t)$ of (1) to the solution of the approximation given by the Finite Volume Method.

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