INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Supp , Pages 100–113

MESH OPTIMIZATION BASED ON THE CENTROIDAL VORONOI TESSELLATION

DESHENG WANG AND QIANG DU

Abstract. The subject of mesh generation and optimization is very important in many scientific applications. In this paper, we investigate the issue of mesh optimization via the construction of Centroidal Voronoi Tessellations. Given some initial Delaunay meshes with only average quality, it is shown that the CVT based mesh optimization generates a robust, high quality mesh which does not rely critically on the choice of the initial mesh. In comparison, other smoothing techniques, such as the classical Laplacian smoothing, tend to be more sensitive to the initial distributions of vertices. Thus, the CVT based optimization may be advocated as a prefered choice for mesh optimization and smoothing.

Key Words. Voronoi tessellations, Delaunay triangulation, optimal tessellations, mesh optimization, mesh smoothing, Centroidal Voronoi tessellation

1. Introduction

The automatic unstructured triangular/tetrahedral mesh generation for complex geometries is essential to the efficient solution of complex problems in various applications such as CFD, CEM and oil reservoir simulations. The advancing front techniques, Octree methods and Voronoi Delaunay-based methods are three well-studied techniques in unstructured mesh generation[1, 2, 3, 4, 5]. Regardless of the method chosen, the resulting unstructured mesh often requires further improvement and optimization. For example, much attention has been paid to the almost regular triangular/tetrahedral meshing used in conjunction with the Yee's scheme in computational electro-magnetics and the MAC method in CFD[37, 38, 39]. Such simulation requirement poses challenges on mesh improvement and optimization, especially in complicated domains.

Traditionally, the procedures for unstructured mesh optimization generally fall into the following basic categories[12, 29, 30, 31, 32, 33, 34, 35]: *geometric optimization*, meaning mesh smoothing or vertices relocation without changing the node connectivity, through strategies such as the Laplacian smoothing and its variants; *topological optimization*, consisting of local reconnections such as edges/faces flipping, while keeping node positions unchanged; and *vertex insertion or deletion*, referring to operations such as the sink insertion[42]. These techniques are often combined and performed in an iterative manner, and they form the core of the *classical optimization methods*. More recently, there have also been some studies on the

2000 *Mathematics Subject Classification.* 65D18, 65N50, 68U07, 65Y20.
Research is supported in part by NSF DMS-0409297 and CCF-0430349. Part of this work was completed while the authors were at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, through the support of the Chinese Academy of Sciences and the China State Major basic research fund G1999032800. Special thanks also to Prof.Jiachang Sun, the guest editor of this special issue, for organizing the SCPI04 international conference.

use of global optimization approaches, such as the use of Winslow transforms, harmonic mappings and algebraic or geometric mesh quality measures [29, 30, 31, 32].

In this paper, we focus on the application of Centroidal Voronoi tessellations (CVTs) to mesh optimizations. The concept of CVT has been used in diverse applications, such as data and image analysis, communication and sensor network, clustering, vector quantization, flow control, dimension reduction and resource allocation[6, 8, 9]. CVTs are defined as special Voronoi tessellations of a region such that the generating points of the tessellations are also the mass centroids of the corresponding Voronoi regions with respect to a given density function[6]. In the application to quality mesh generation, a CVT configuration provides an optimal points distribution (with respect to a given density), its dual centroidal Voronoi-Delaunay triangulation (CVDT) provides a high quality triangular (or tetrahedral) mesh[7, 12]. The optimality can be illustrated through the minimization of an associated error or cost functional, and it can also be validated by the celebrated Gersho's conjecture which predicts the asymptotic equi-partition of the local error. CVTs can often be constructed through the iterative Lloyd algorithm which moves the generators to the mesh centers and re-start the Voronoi-Delaunay construction. Thus, if Lloyd iteration is applied to an initial Delaunay triangular mesh to construct a CVDT or a constrained CVDT of a given domain, the final triangular mesh becomes a natural optimization of the initial mesh. CVT based mesh optimization has been successfully applied to 2D/3D isotropic cases [7, 12, 16], and it has also been generalized to anisotropic and surface mesh generation [10, 15]. A brief survey can be found in [18].

Some earlier results reported on the CVT based mesh optimization show encouraging signs that it may be further developed into a robust procedure for improving the mesh quality. In this paper, we carry out more numerical studies on the effectiveness of its applications to the isotropic 2D and 3D mesh optimization and also make comparisons with other existing algorithms. For two dimensional examples, the Lloyd iterations with respect to the constant density yield meshes that are almost regular triangular meshes. The comparisons between the classical optimization techniques that combine mesh smoothing with edges/faces swapping and the CVT based optimization technique indicate that the classical optimization is much more sensitive to the initial mesh configuration or vertex distribution, while the CVT based optimization provide meshes that are largely independent of such initial conditions. Similarly, for the three dimensional application examples, we can also see that the CVT based optimization results in meshes that are of higher quality and are more structured than those obtained by the classical optimization.

The remaining part of the paper is organized as follows. The basic procedures of the mesh optimization based on the centroidal Voronoi tessellation are recalled in Section 2. The effects of the mesh improvement based on the CVT and comparisons with those of classical optimizations are discussed in Section3 and Section4, for 2D and 3D isotropic meshing respectively. A final conclusion is made in Section5.

2. Mesh Optimization Based on Centroidal Voronoi Tessellation

Recently, the centroidal Voronoi tessellation (CVT) and its wide range of applications have been studied in [6, 7, 8, 9, 10, 11, 12]. Often, CVT provides optimal points placement with respect to a given density function. When the density function is chosen properly with respect to a giving sizing field, its dual structure, the so-called centroidal Voronoi Delaunay triangulation (CVDT), results in a high-quality Delaunay mesh[7, 12]. We have applied this technique to mesh generation