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ON TWO ITERATION METHODS FOR THE QUADRATIC MATRIX EQUATIONS

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Abstract. By simply transforming the quadratic matrix equation into an equivalent fixed-point equation, we construct a successive approximation method and a Newton's method based on this fixed-point equation. Under suitable conditions, we prove the local convergence of these two methods, as well as the linear convergence speed of the successive approximation method and the quadratic convergence speed of the Newton's method. Numerical results show that these new methods are accurate and effective when they are used to solve the quadratic matrix equation.

Key Words. Quadratic matrix equation, iteration method, convergence property.

1. Introduction

The *quadratic matrix equation* (**QME)**

$$\mathcal{Q}(X) \equiv X^2 - BX - C = 0, \quad B, C \in \mathbb{C}^{n\times n} \tag{1}$$

occurs in a variety of applications. For example, it may arise in the quadratic eigenvalue problem[3, 4, 6, 8, 12, 13]

$$\mathcal{Q}(\lambda)x \equiv \lambda^2 x - \lambda Bx - Cx = 0, \quad B, C \in \mathbb{C}^{n\times n},$$

or the noisy Wiener-Hopf problems for Markov chains[5, 7, 10, 11]. Evidently, some Riccati equations are QMEs, and vice versa, and theory of Riccati equations and numerical methods for their solution are well developed[2, 9]; however, these two classes of equations require different techniques for analysis and solution in general. See also [1].

Recently, Higham and Kim[6] studied Newton's methods with and without exact line searches for solving the QME(1). In the Newton's method, the quadratic matrix function $\mathcal{Q}(X)$ is successively linearized at each of the current iterate $X^{(k)}$ which is required to be located in a neighborhood of a solution $X_\star$ of the QME(1), and the next iterate $X^{(k+1)}$ is obtained by solving the corresponding Newton equation which is a special case of the generalized Sylvester equation. And in the Newton's method with line search, the current Newton direction $E^{(k)}$ is used as a search direction and the next iterate

$$X^{(k+1)} = X^{(k)} + t^{(k)}E^{(k)}$$

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is defined by exactly minimizing the objective function

$$p(t) = \|\mathcal{Q}(X^{(k)} + t^{(k)}E^{(k)})\|_F^2$$

along this direction, i.e.,

$$t^{(k)} = \mathrm{argmin}_{0<t<2}\, p(t),$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. It was proved in [6] that the latter has global convergence property.

In particular, when B is a diagonal matrix and C is an M-matrix, Guo[5] studied the existence and uniqueness of M-matrix solutions and iterative method for finding the desired M-matrix solution of the QME(1) by transforming it into a special nonsymmetric *algebraic Riccati equation* **(ARE)**, and proved the monotone convergence of the obtained iterative methods.

In this paper, for general matrices $B, C \in \mathbb{C}^{n\times n}$, we first simply transform the QME(1) into an equivalent fixed-point equation, and then based on it we construct a successive approximation method and a Newton's method for solving the quadratic matrix equation (1). Under suitable conditions, we prove the local convergence of these two methods, as well as the linear convergence speed of the successive approximation method and the quadratic convergence speed of the Newton's method. Numerical results show that these new methods are more accurate and effective than the known ones in [6, 5].

Without loss of generality, throughout this paper we will assume that the constant matrix term $C \in \mathbb{C}^{n\times n}$ in the QME(1) is nonsingular. In the case that the matrix C is singular, we can shift the variable and make the constant matrix term in the equivalently transformed quadratic matrix equation be nonsingular. More specifically, by letting $Y = \sigma I - X$ we can rewrite the QME(1) as

$$Y^2 - (2\sigma I - B)Y + (\sigma^2 I - \sigma B - C) = 0,$$

where σ is a real constant. We can now choose the parameter σ such that the matrix $(\sigma^2 I - \sigma B - C)$ is nonsingular. See [5].

2. Two iteration methods

If $X_\star \in \mathbb{C}^{n\times n}$ is a solution of the QME(1), i.e.,

$$\mathcal{Q}(X_\star) = X_\star^2 - BX_\star - C = 0,$$

then we have

$$(X_\star - B)X_\star = C.$$

It then follows that both $X_\star$ and $(X_\star - B)$ are nonsingular matrices, provided C is a nonsingular matrix. In this case, we can construct the following fixed-point equation for the QME(1):

$$X = \mathcal{F}(X), \quad \text{where} \quad \mathcal{F}(X) = (X - B)^{-1}C. \tag{2}$$

Therefore, $X_\star \in \mathbb{C}^{n\times n}$ is a solution of the QME(1) if and only if it is a fixed-point of the matrix operator $\mathcal{F}(X)$, or equivalently, a zero point of the matrix equation

$$X - \mathcal{F}(X) = 0.$$

Furthermore, by denoting

$$\mathcal{G}(X) = X - \mathcal{F}(X)$$

and using the first-order approximation to $\mathcal{G}(X)$, we have

$$\mathcal{G}(X + E) = \mathcal{G}(X) + \mathcal{J}(X, E) + \mathcal{O}(E^2),$$