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# PRECONDITIONED HYBRID CONJUGATE GRADIENT ALGORITHM FOR P-LAPLACIAN

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**Abstract.** In this paper, a hybrid conjugate gradient algorithm with weighted preconditioner is proposed. The algorithm can efficiently solve the minimizing problem of general function deriving from finite element discretization of the p-Laplacian. The algorithm is efficient, and its convergence rate is mesh-independent. Numerical experiments show that the hybrid conjugate gradient direction of the algorithm is superior to the steepest descent one when $p$ is large.

**Key Words.** p-Laplacian, finite element approximation, hybrid conjugate gradient algorithm, numerical experiments

## 1. Introduction

Let $\Omega$ be a bounded open subset of $R^2$ with a Lipschitz boundary $\partial\Omega$. The p-Laplacian with Dirichlet data is the following equation (1.1):

$$\begin{aligned} -div(|\bigtriangledown u|^{p-2} \bigtriangledown u) &= f, \ in \ \Omega \\ u &= 0, \ on \ \partial\Omega \end{aligned}$$

where $1 < p < \infty$, $f \in L^2(\Omega)$, and $|\cdot|^2 = (\cdot,\cdot)_{R^2}$.

When $p = 2$, the equation (1.1) becomes a linear Laplacian equation. The equation (1.1) occurs in many mathematical models of physical process, for instances, glaciology, nonlinear diffusion and filtration(see Philip [21]), power-law materials(Atkinson and Champion [2]), and quasi-Newtonian flows(Atkinson and Jones [3]). The equation (1.1) is viewed as one of the typical examples of a large class of nonlinear problems. It contains most of the essential difficulties in studies of finite element approximations for this class of degenerate nonlinear systems. For this class of systems, many existing techniques in the finite element method, for example, the linearization method and deformation procedure, do not seem to work well.

Finite element approximations of p-Laplacian have been extensively studied in the literature, for example, in [10, 1, 12, 7, 8, 20]. In particular, the quasi-norm approach has proved quite successful in deriving sharp a priori and a posteriori error bounds for the finite element approximation of the degenerate systems. A priori and a posteriori error bounds for p-Laplacian are proposed by using quasi-norm approach in the paper [14, 15, 16].

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Solving the equation (1.1) is equivalent to solve the following minimization problem:

$$\min_{v\in V} J(v) \tag{1.2}$$

where $V = W_0^{1,p}(\Omega)$,$1 < p < \infty$,and

$$J(v) = \frac{1}{p}\int_\Omega |\bigtriangledown v|^p - \int_\Omega fv \tag{1.3}$$

Huang, Li and Liu[13] proposed a steepest descent algorithm with weighted preconditioner which is solved by an algbric multigrid method. The decent algorithm has excellent computing efficiency for both $p$ large or relatively small, for example, $p = 1000$ and $p = 1.5$, which are obviously superior to past methods. Tai and Xu[22] proposed a pure multigrid algorithm for solving the nonlinear problems including the p-Laplacian. Some theoretical and numerical analysis show the good efficiency.

It is well known that the conjugate gradients or their hybrid algorithms are more efficient than the steepest descent algorithm when solving nonlinear programming. Based on this thought, we proposed a hybrid conjugate gradient algorithm with weighted preconditioner in this paper. The new algorithm is more efficient than the descent one in the paper [13] for p-Laplacian for large $p$. The paper is organized as follows. Section 2 is devoted to mathematical preliminaries. In Section 3, we propose the hybrid conjugate gradient algorithm with weighted preconditioner. In Section 4, we present numerical results in order to compare and evaluate the performance of the new method and the steepest descent algorithm, and finally end, in Section 5, with some conclusions and discussions.

## 2. Preliminaries

Obviously, the functional $J(v)$ decided by (1.3) is strictly convex for $1 < p < \infty$. Furthermore, the equation (1.2) has a unique solution. It is well known that solving the equation (1.2) is equivalent to the following nonlinear PDE-the p-Laplacian:

$$(WP) \quad a(u,v) = \int_\Omega |\bigtriangledown u|^{p-2} \bigtriangledown u \bigtriangledown v = \int_\Omega fv, \quad \forall v \in V. \tag{2.1}$$

A direct calculation yields

$$J'(u)(v) = \int_\Omega |\bigtriangledown u|^{p-2} \bigtriangledown u \bigtriangledown v - \int_\Omega fv. \tag{2.2}$$

One can refer to the paper [9] for other conclusions of $J'(u)(v)$ and $J''(u)(v,w)$. We now introduce the finite element spaces. Let $T^h$ be a regular triangulation of $\Omega^h$, which is composed of disjoint open regular triangles $K_i$, that is , $\bar{\Omega}^h = \bigcup_{K_k\in T^h} \bar{K}_i$, where $h = \max_{K\in T^h} h_k$, and $h_k$ denotes the diameter of the element $K$ in $T^h$. When $i \neq j$, $\bar{K}_i \bigcap \bar{K}_j$ is void, or only one common vertex, or a whole edge.

Because of the limited higher order regularity for the solution of the p-Laplacian (see [2, 3, 22]), we shall only discuss the continuous piecewise linear element in this paper. Associated with $T^h$ is a finite dimensional subspace $V^h$ of $C^0(\bar{\Omega}^h)$, such that $\chi|_K \in \mathcal{P}_1$ for all$\chi \in V^h$ and $K \in T^h$, where $\mathcal{P}_1$ is the linear function space. Let

$$V_0^h = \{\chi \in V^h : \chi(x^k) = 0,\ for\ all\ x^k \in \partial\Omega^h\}$$

Then the finite element approximation of $(WP)$ is as follows $(WP)^h$: Find $u_n \in V_0^h$ such that

$$(WP)^h \quad \int_{\Omega^h} |\bigtriangledown u_n|^{p-2} \bigtriangledown u_n \bigtriangledown v_n = \int_{\Omega^h} fv_h \tag{2.3}$$