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ON A ROBUST ITERATIVE METHOD FOR HETEROGENEOUS HELMHOLTZ PROBLEMS FOR GEOPHYSICS APPLICATIONS

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Abstract. In this paper, a robust iterative method for the 2D heterogeneous Helmholtz equation is discussed. Two important ingredients of the method are evaluated, namely the Krylov subspace iterative methods and multigrid based preconditioners. For the Krylov subspace methods we evaluate GMRES and Bi-CGSTAB. The preconditioner used is the complex shifted Laplace preconditioner [*Erlangga, Vuik, Oosterlee, Appl. Numer. Math. 50(2004) 409–425*] which is approximately solved using multigrid. Numerical examples which mimic geophysical applications are presented.

Key Words. Helmholtz equation, Krylov subspace methods, preconditioner, multigrid

1. Introduction

Wave equation migration is becoming increasingly popular in seismic applications. This migration is currently based on a one-way scheme to allow applications in 3D, in which the full wave equation simulation is simply too expensive. It is already known, however, that one-way wave equations do not correctly image steep events and do not accurately predict the amplitudes of the reflections [12].

In 2D, the linear system obtained from the discretization of the full wave equation in the frequency domain can be efficiently solved with a direct solver and a nested dissection ordering [6]. In 3D, the band size of the linear system becomes too large, which makes the direct method inefficient. As an alternative, iterative methods can be used.

Since 3D problems are our final goal, iterative methods become inevitable. In this paper an evaluation of a robust iterative solver for Helmholtz problems is discussed. The solver mainly consists of two important ingredients: Krylov subspace iterative methods, and a preconditioner including multigrid to accelerate the Krylov subspace iterations.

Krylov subspace methods are chosen because the methods are efficient in terms of memory requirement as compared to direct solvers. Multigrid is used as preconditioner for the Krylov subspace methods. In our applications, however, multigrid is not directly applied to the Helmholtz equation. As already pointed out in [3], high wavenumber problems related to the Helmholtz equation raise difficulties for multigrid in both error smoothing and coarse grid correction, the two main principles of multigrid. Instead, we use multigrid on a Helmholtz-like preconditioner that multigrid can handle it easily. In particular, we consider a Helmholtz operator

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with a complex shift. An operator-based preconditioner for the Helmholtz equation is first proposed by Bayliss et. al [1] in the early eighties and solved with multigrid in [8]. Laird and Giles [10] proposed a real positive definite Helmholtz operator (i.e. the same Helmholtz operator but with sign reverse for the zeroth order term) as the preconditioner. Our preconditioner [5] is a complex version of a Helmholtz operator.

This paper is organized as follows. In §2, the Helmholtz equation and preconditioners for iteratively solving it are discussed. Some properties of the preconditioned linear system are explained in §3. Multigrid is briefly discussed in §5. We present numerical examples and some conclusions in §6 and §7, respectively.

2. Helmholtz equation, preconditioner

For a given source function g, we are interested in the solution of the Helmholtz equation

$$\mathcal{A}\phi := -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\phi - (1-\alpha i)k^2\phi = g, \text{ in } \Omega \subset \mathbb{R}^d,\ d = 1,2,3, \tag{1}$$

which governs wave propagations in the frequency domain. Here, $\phi = \phi(x_1, x_2, x_3) \in \mathbb{C}$ is usually the pressure wave, and k, the wavenumber, varies in Ω due to spatial variation of local speed of sound, c. This wavenumber is defined as $k = \omega/c$, where ω is the angular frequency related to the source function g. We call the medium "barely attenuative" if $0 < \alpha \ll 1$. In (1), $i = \sqrt{-1}$, the complex identity.

Boundary conditions on $\Gamma = \partial\Omega$ are usually in the form of absorbing boundary condition. There are several mathematical representations to satisfy this condition. In [4] hierarchical, local boundary conditions are proposed. A perfectly matched layer can also be used to ensure absorbing boundary (see [2]). In this paper we use two types of the hierarchical absorbing boundary conditions: (i) the first order formulation, namely

$$\mathcal{B}_1\phi := \frac{\partial\phi}{\partial\nu} - ik\phi = 0, \qquad \text{on } \Gamma \tag{2}$$

with ν the outward normal direction to the boundary, and (ii) the second order formulation

$$\mathcal{B}_2\phi := \frac{\partial\phi}{\partial\nu} - ik\phi - \frac{i}{2k}\frac{\partial^2\phi}{\partial\tau^2} = 0, \tag{3}$$

with τ the tangential direction. The second order absorbing condition is more accurate in handling inclined outgoing waves at the boundary than the first order boundary condition, but it requires careful implementation.

Discretization of (1) using finite differences/elements/volumes leads to an indefinite linear system

$$\boldsymbol{A\phi} = \boldsymbol{g} \tag{4}$$

for large wavenumbers. We use a 5-point finite difference approximation to (1) and (2) (or (3)). Furthermore, only for sufficiently small k the problem is definite. For definite elliptic problems, preconditioned Krylov subspace methods and multigrid are two examples of good solvers and have been widely used. For the Helmholtz equation, both methods, however, are found to be less effective, or even ineffective, if k is large.

For Krylov subspace methods, the methods usually suffer from slow convergence. In this kind of situation the methods rely on preconditioners. Finding good preconditioners for the Helmholtz equation, however, is not a trivial task. Since $\boldsymbol{A}$