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OPTIMAL UNIFORM CONVERGENCE ANALYSIS FOR A TWO-DIMENSIONAL PARABOLIC PROBLEM WITH TWO SMALL PARAMETERS

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Abstract. In this paper, we consider a two-dimensional parabolic equation with two small parameters. These small parameters make the underlying problem containing multiple scales over the whole problem domain. By using the maximum principle with carefully chosen barrier functions, we obtain the pointwise derivative estimates of arbitrary order, from which an anisotropic mesh is constructed. This mesh uses very finer mesh inside the small scale regions (where the boundary layers are located) than elsewhere (large scale regions). A fully discrete backward difference Galerkin scheme based on this mesh with arbitrary k-th ($k \geq 1$) order conforming rectangular elements is discussed. Note that the standard finite element analysis technique can not be used directly for such highly nonuniform anisotropic meshes because of the violation of the quasi-uniformity assumption. Then we use the integral identity superconvergence technique to prove the optimal uniform convergence $O(N^{-(k+1)} + M^{-1})$ in the discrete L^2-norm, where N and M are the number of partitions in the spatial (same in both the x- and y-directions) and time directions, respectively.

Key Words. Singular perturbation, anisotropic mesh and uniform convergence.

1. Introduction

Singular perturbation problems (SPPs) appear in many areas, such as in chemical kinetics, heterogeneous flow in porous media, periodic structures, and plate and shell problems, etc. Actually, "Such a situation is relatively common in applications, and this is one of the reasons that perturbation methods are a cornerstone of applied mathematics" [16, Preface]. Those small parameters make the underlying problems contain multiple scales spanning over the whole domain. It is well known that the solutions of singular perturbation problems usually undergo rapid changes within very thin layers near the boundary (boundary layers) or inside the problem domain (interior layers), where the small scales are located.

However, direct numerical simulation by using the standard finite element method to resolve such multiscale problems is very impractical due to the requirement of

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huge computer memory and CPU time. For example, by using a linear finite element on a quasi-uniform mesh to solve the simple model

$$-\varepsilon^2 \triangle u + u = f(x,y) \quad \text{in } \Omega \subseteq \mathcal{R}^2, \quad u|_{\partial\Omega} = 0,$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, we can obtain the following global error estimate:

$$||u - u_h||_\varepsilon \le C(\varepsilon + h)h||u||_{H^2(\Omega)},$$

where $||u||_\varepsilon = (\varepsilon^2||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)})^{1/2}$. Noticing the fact that [31, Lemma 2.1]:

$$||u||_{H^2(\Omega)} \le C\varepsilon^{-2}||f||_{L^2(\Omega)}, \tag{1}$$

we see that, to ensure good approximation, the mesh size h must be in the order of $o(\varepsilon)$. Suppose that $\varepsilon = 10^{-6}$ (which is very common), then $h = o(10^{-6})$. Hence we will end up with 10^{12} unknowns, which is well out of the power of most today's computer resources.

In summary, solving SPPs is a very challenging task because of the fact that ε can be very small leads to notorious computational difficulties [25, pp.310]. Such difficulties have also been emphasized by many researchers [30, 11]. The challenging SPPs serve frequently as test models for new algorithms, e.g., in multigrid methods [14, Ch.10], domain decomposition methods [12], collocation methods [4, Ch.10], and adaptive methods [1, 32].

Recently, the standard finite element methods based on anisotropically refined meshes, which use different scales of mesh size in different subdomains, were proved to give uniform convergence, which is independent of the small perturbation parameters. However, most work was restricted to linear finite element and problems with one perturbation parameter [3, 19, 22, 29, 37]. More details about the unsolved problems in this area can be found in the most recent survey by Roos [28].

In this paper, we will consider the analysis of applying arbitrary order tensor-product finite elements on such highly nonuniform anisotropic mesh to a two-dimensional parabolic equation with two small parameters. The pointwise derivative estimates are essential in the construction of such an anisotropic mesh with optimal uniform convergence. Here we use the maximum principle [26] as our powerful tool to obtain those derivative estimates by carefully choosing all kinds of barrier functions. Then we use the integral identity superconvergence technique [23, 7, 37, 9] originally developed for superconvergence analysis on tensor-product finite elements. We like to remark that uniform convergence can not be obtained directly by the standard finite element analysis for such highly nonuniform anisotropic meshes because of the violation of the quasi-uniformity assumption [8, 5]. Special interpolation estimates have to be obtained on such anisotropic meshes [2]. Also asymptotical expansion or pointwise derivative estimates for the analytical solution has to be investigated in order to obtain such uniform convergence [21, 22].

For simplicity, here we focus on the following parabolic equation

$$L_{\varepsilon\mu} u \equiv \varepsilon\frac{\partial u}{\partial t} - \mu^2 a \triangle u + bu = f(x,y,t,\varepsilon,\mu) \quad \text{in } D \equiv \Omega \times (0,T], \tag{2}$$

$$u|_{\partial\Omega\times(0,T]} = 0, \quad u|_{t=0} = 0, \tag{3}$$

where $\Omega = (0,1)^2$, and the coefficients $a(x,y,t), b(x,y,t)$ and f are sufficiently smooth functions. Here $0 < \varepsilon \ll 1, 0 < \mu \ll 1$ are small parameters. Furthermore