INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 1, Pages 19–30

LOCATING NATURAL SUPERCONVERGENT POINTS OF FINITE ELEMENT METHODS IN 3D

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(Communicated by Yanping Lin)

Abstract. In [20], we analytically identified natural superconvergent points of function values and gradients for several popular three-dimensional polynomial finite elements via an orthogonal decomposition. This paper focuses on the detailed process for determining the superconvergent points of pentahedral and tetrahedral elements.

Key Words. Finite element methods, three-dimensional problems, natural superconvergence, pentahedral elements and tetrahedral elements.

1. Introduction

Superconvergence of the finite element (FE) method is a phenomenon that, at special *a priori* points, the convergent rate of FE approximations exceeds what is globally possible. By *natural superconvergence*, we mean that a higher order accuracy is achieved without applying any recovering or averaging techniques in the FE solution.

There have been many studies concerning with superconvergence of FE methods since the 1970s [8]. Books and survey papers have been published. For the literature, we refer to [1, 6, 7, 11, 12, 13, 17, 21] and references therein.

In [3], Babuška *et al.* predicted derivative superconvergent points for the Laplace equation, the Poisson equation, and linear elasticity equations. They reduced the problem of finding natural superconvergent points to the problem of finding intersections of certain polynomial contours. The actual superconvergent points were determined by computer programs without explicitly constructing those polynomials. Therefore, this approach is called *"computer-based"* proof.

Later, Zhang proposed an analytic approach. By an orthogonal decomposition under local rectangular and brick (hexahedral) meshes [18, 19], he constructed explicitly the polynomials for determining the superconvergent points in the "computer-based" proof and obtained analytically, superconvergence results in FE solutions for the tensor product, serendipity, and *intermediate* families. In [14], the authors studied natural superconvergence of derivatives and function values under

Received by the editors July 1, 2004 and, accepted for publication October 10, 2004.
2000 *Mathematics Subject Classification.* 65N30, 65N15.
This research was supported by the US National Science Foundation grants DMS-0311807.

four mesh patterns of triangular elements via the same approach. Our results confirmed those provided in [3] by the "computer-based" proof. Moreover, many new superconvergence results were obtained.

Recently, the authors have reported some natural superconvergence results for several 3D FE, which are used to approximate sufficiently smooth solutions of the Poisson and the Laplace equations [20]. The main theorems in the "computer-based" proof are generated to the 3D problems. Several FE meshes and spaces are studied in details. In particular, Lagrangian and serendipity elements of both hexahedron and pentahedron (triangular prism) are considered. Two patterns of tetrahedral elements are also discussed.

We notice that there are many cases involved in the investigation [20], and the process in 3D problems is rather complicated. Therefore, detailed explanation is necessary for retrieval and a better recognition of the method. Since the situation for the hexahedral elements has been thoroughly studied in [18, 20], the present paper shall focus on the approach of locating superconvergent points of pentahedral and tetrahedral elements.

In Section 2, some notations are introduced. The FE meshes and spaces are also described. The procedure of how to determine superconvergent points are illustrated through detailed examples. In particular, an example for tetrahedral element is provided in Section 3. Several points of comparison between two tetrahedral partitions are also addressed. Examples for the Lagrangian and serendipity pentahedral elements are given in Section 4 and 5, respectively. A summary for superconvergence results of the discussed elements of order 2 and 3 is given in the last section.

2. Preliminaries

2.1. Notations. Assume that a 3D FE mesh is locally translation invariant. Then, as shown in [3, 20], the task of finding superconvergent points can be narrowed down in the *master cell*, or equivalently in the *reference cell* $K = [-1, 1]^3$. In the context, (ξ, η, ζ) is used for the standard Euclidean coordinates in K.

Let $V_n(K)$ and $V_n^\pi(K)$ be the *FE local space* and the *periodic FE local space* of order n defined on K, respectively. Let Π_n be the space defined by

$$\Pi_n = \mathrm{Span}\, \left\{ \xi^i \eta^j \zeta^k \,|\, 0 \le i, j, k \le n,\, 0 \le i + j + k \le n \right\}.$$

Write $\Pi_n(K)$ the restriction of Π_n on K. Associated with a particular partition of K, we denote $\Pi_n^w(K)$ (resp. $\Pi_n^\pi(K)$) the set of *piecewisely continuous* (resp. *periodic piecewise continuous*) polynomials of degree not greater than n.

We need to point out that, for pentahedral elements, $\Pi_n^w(K)$ is a proper subset of $V_n(K)$, and $\Pi_n^\pi(K) \subsetneq V_n^\pi(K)$; For tetrahedral elements, $\Pi_n^w(K) = V_n(K)$, and $\Pi_n^\pi(K) = V_n^\pi(K)$.

Denote the set of $(n+1)$th degree *homogeneous harmonic polynomials* in three variables by $\mathcal{H}_{n+1}$. Then it is shown that $\dim \mathcal{H}_{n+1} = 2n + 3$ [2]. Furthermore, by the *Kelvin transform*, we can obtain an explicit basis of $\mathcal{H}_{n+1}$.