INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 1, Pages 43–56

SUPERCONVERGENCE PHENOMENA ON THREE-DIMENSIONAL MESHES

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(Communicated by Zhimin Zhang)

Abstract. We give an overview of superconvergence phenomena in the finite element method for solving three-dimensional problems, in particular, for elliptic boundary value problems of second order over uniform meshes. Some difficulties with superconvergence on tetrahedral meshes are presented as well. For a given positive integer m we prove that there is no tetrahedralization of R^3 whose all edges are m-valent.

Key Words. Linear and quadratic tetrahedral elements, acute partitions, Poisson equation, postprocessing, supercloseness, averaging and smoothing operators, regular polytopes, combinatorial topology.

1. Introduction

In 1966 Babuška, Práger, and Vitásek (see [2, Sect. 4.3]) developed a special finite difference method for the equation

$$-(pu')' + qu = f \text{ in } (0, 1)$$

with mixed boundary conditions. Using the Marchuk identities and sophisticated numerical quadrature rules, they obtained a numerical scheme yielding the accuracy $\mathcal{O}(h^6)$ at nodal points. The associated system of linear algebraic equations has only a tridiagonal matrix (like for linear finite elements). In 1972 Douglas and Dupont (see [16]) called (for the first time) a similar high order accuracy phenomenon in the finite element method *superconvergence*. Some very early finite element superconvergence results from the period 1966–1969 are mentioned in [64, p. vi]. Surveys on other superconvergence phenomena can be found, e.g., in [10], [12], [13], [14], [27], [30], [31], [32], [42], [64], [71], [72]. Superconvergence is a useful tool in a posteriori error estimations, mesh refinement and adaptivity. At present, the total number of papers on superconvergence is about 1000.

A key assumption in proving many superconvergence phenomena is a high regularity of the exact solution and also some regular structure of the partitions used (uniform, piecewise uniform, locally quasiuniform, locally periodic, locally point-symmetric, self-similar etc.). Throughout this paper we shall use standard face-to-face partitions into elements in R^d, $d \in \{1, 2, 3, \dots\}$. Also we shall only consider

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Received by the editors July 3, 2004 and, accepted for publication in revised form October 30, 2004.

2000 *Mathematics Subject Classification.* 65N30, 51M20.

This research was supported by grant no. 201041503 of the Grant Agency of the Czech Republic.

regular families of partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$, i.e., there exists a constant $C > 0$ such that for all elements $T \in \mathcal{T}_h$ and all partitions $\mathcal{T}_h \in \mathcal{F}$ we have

$$\text{meas}_d T \geq C h_T^d,$$

where h_T is the diameter of T and meas_d stands for the d-dimensional Lebesgue measure.

In the next section we deal with superconvergence of linear elements on tetrahedral meshes for solving the Poisson equation with Dirichlet boundary conditions. The main idea is based on the fact that the gradient of the Ritz-Galerkin solution is superclose to the gradient of the Lagrange interpolation when uniform meshes are employed.

In the third section we recall some superconvergence phenomena for standard quadratic tetrahedral elements, which are frequently used in applications. These phenomena are based on some special properties of an important subclass of basis functions, namely piecewise quadratic bubble functions.

In the fourth section we give an overview of some other superconvergence phenomena for the finite element method in three-dimensional space. In particular, we present superconvergence results obtained for the solution and its gradient of second order boundary value problems of elliptic type when using rectangular trilinear and triquadratic elements, Serendipity elements, etc.

Finally, in the last section we present an unusual difficulty with superconvergence on three-dimensional meshes, which does not arise in solving two-dimensional problems. Namely, one can prove $\mathcal{O}(h^4)$-superconvergence at nodal points when solving the Poisson equation by linear elements over triangulations consisting solely of equilateral triangles. Such a result cannot be generalized to three-dimensional space, since the regular tetrahedron is not a space-filler.

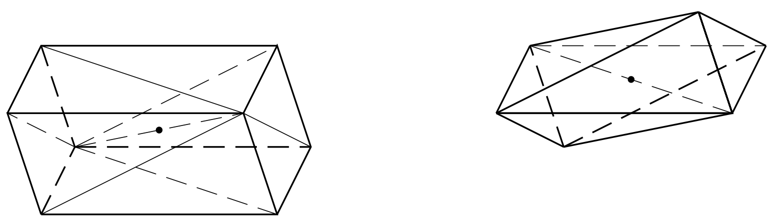
2. Superconvergence of linear elements on tetrahedral meshes

Let $\Omega \subset R^d$ be a bounded polytopic (polyhedral for $d = 3$) domain with Lipschitz boundary. We shall use the standard Sobolev space notation of norms and seminorms. For simplicity, let us consider only the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \tag{1}$$

with homogeneous Dirichlet boundary conditions.

By a *tetrahedralization* (finite or infinite) we shall mean any face-to-face partition of a polyhedron or R^3 into closed tetrahedra. In this and the next section, we shall only consider so-called *uniform tetrahedralizations of* $\overline{\Omega}$, i.e., for each internal edge $\ell \not\subset \partial\Omega$ the patch of tetrahedra sharing ℓ is a point-symmetric set with respect to the midpoint M of ℓ (see Figure 1).



We shall look for $u_h \in V_h$ such that $(\nabla u_h, \nabla v_h)_0 = (f, v_h)_0$ for all $v_h \in V_h$, where $V_h \subset H_0^1(\Omega)$ is the space of continuous piecewise linear functions over a given tetrahedralization $\mathcal{T}_h$. In 1969 Oganesjan and Ruhovec (see [49]) proved for linear triangular elements over uniform partitions (i.e., when any two adjacent triangles