INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 1, Pages 57–74

CONVERGENCE ANALYSIS OF FINITE ELEMENT SOLUTION OF ONE-DIMENSIONAL SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS ON EQUIDISTRIBUTING MESHES

WEIZHANG HUANG

(Communicated by Zhimin Zhang)

Abstract. In this paper convergence on equidistributing meshes is investigated. Equidistributing meshes, or more generally approximate equidistributing meshes, are constructed through the well-known equidistribution principle and a so-called adaptation (or monitor) function which is defined based on estimates on interpolation error for polynomial preserving operators. Detailed convergence analysis is given for finite element solution of singularly perturbed two-point boundary value problems without turning points. Illustrative numerical results are given for a convection-diffusion problem and a reaction-diffusion problem.

Key Words. Mesh adaptation, equidistribution, error analysis, finite element method.

1. Introduction

The concept of equidistribution has been used for long for adaptive mesh generation. It is first used by Burchard [7] and then by a number of researchers (cf. the early works [2, 13, 15, 24, 27]) for the error analysis of best spline approximations with variable knots. An algorithm, now known as de Boor's algorithm, is introduced by de Boor [14] for computing equidistributed meshes. Russell and Christiansen [31] give an early review on mesh selection strategies based on equidistribution, and one of such strategies is implemented in the general-purpose code COLSYS by Ascher, Christiansen, and Russell [1]. The equidistribution principle has also been playing an important role in multi-dimensional adaptive mesh generation. The concept can naturally be incorporated into the variational mesh generation framework, and a number of methods have been developed along this line, e.g., see [6, 8, 9, 16, 17, 19, 21].

Convergence analysis for the numerical solution of partial differential equations (PDEs) using equidistributing meshes can be traced back to works in the seventies of the last century. For example, Pereyra and Sewell [27] give an asymptotical bound for the truncation error when finite differences are used for solving two-point boundary value problems on an equidistributing mesh. Babuška and Rheinboldt [2] obtain a posteriori error estimates for finite element solutions for one dimensional

Received by the editors January 1, 2004 and, in revised form, March 22, 2004.
2000 *Mathematics Subject Classification.* 65M50, 65M60, 65L50, 65L60.
This work was supported in part by the NSF under grants DMS-0074240 and DMS-0410545 and by the University of Kansas General Research Fund under grant #2896 (2004).

problems in an asymptotic form for the size of elements going to zero. They show that a mesh is asymptotically optimal if all error indicators on subintervals are equal (and thus the mesh is equidistributing). Recent progress has been made on mathematically more rigorous convergence analysis. Notably, Qiu and Sloan [28] and Qiu, Sloan, and Tang [29] investigate the uniform convergence of upwind finite difference approximations to a singularly perturbed problem. Beckett and Mackenzie study the convergence of finite difference approximations to convection-diffusion problems without turning points and reaction-diffusion problems in [3, 4, 23] and finite element approximations to reaction-diffusion problems in [5]. The meshes considered in these works are either chosen a prior or determined through the equidistribution relation based on the explicit expression of the exact solution. The stability and convergence of the finite element solution to one-dimensional convection-dominated problems are studied by Chen and Xu [10] for some a prior chosen meshes. The convergence analysis to a fully discrete problem where meshes are determined completely by the computed solution is recently presented by Kopteva and Stynes [22] for the upwind finite difference discretization of a quasi-linear one-dimensional convection-diffusion problem without turning points.

The objective of this paper is to study the convergence of finite element solution to one-dimensional singularly perturbed PDEs on equidistributing meshes. We attempt to develop a general theory for use in convergence analysis. Our approach is different from those used in the existing works. Specifically, following [18, 20] we first investigate interpolation error for polynomial preserving operators on a general mesh. Several mesh quality measures are defined, and estimates for interpolation error are obtained in terms of these mesh quality measures. An equididistributing mesh which satisfies the equidistribution principle, or more generally, an approximate equididistributing mesh which satisfies the equidistribution principle only approximately, is characterized as a mesh with a bounded overall quality measure (see (16)) as it is refined. The interpolation error estimates are then used to analyze the convergence of the (standard) finite element solution to singularly perturbed PDEs on (approximate) equidistributing meshes. The analysis is carried out for two separate cases, the convection-diffusion case and the reaction-diffusion one. To our best knowledge, this is the first work on the convergence of standard finite element solution of one-dimensional convection-diffusion problems on equidistributing meshes while an analysis for one-dimensional reaction-diffusion problems is given by Beckett and Mackenzie [5]. It is emphasized that unlike the existing approaches, our analysis does not use an a prior chosen mesh nor requires the mesh be given through the equidistribution principle with an analytical expression of the exact solution. What we require is that the mesh satisfies the equidistribution relation approximately, i.e., (16) or (39) and (40). Numerical results presented in Section 4 show that such a mesh can be obtained using De Boor's algorithm [14].

An outline of the paper is as follows. In section 2, we study approximation properties of polynomial preserving operators on a general mesh and define several mesh quality measures. In Section 3, the results of Section 2 are applied to the convergence analysis of the finite element solution to singularly perturbed boundary value problems without turning points. Convection-diffusion and reaction-diffusion equations are covered in Subsections 3.1 and 3.2, respectively. Numerical results are presented in Section 4. Finally, Section 5 contains the conclusions.

Throughout the paper, we use C as a generic constant which may take different values at different occurrences.