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A HIGH ORDER PARALLEL METHOD FOR TIME DISCRETIZATION OF PARABOLIC TYPE EQUATIONS BASED ON LAPLACE TRANSFORMATION AND QUADRATURE

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Abstract. We consider the discretization in time of a parabolic equation, using a representation of the solution as an integral along a smooth curve in the complex left half plane. The integral is then evaluated to high accuracy by a quadrature rule. This reduces the problem to a finite set of elliptic equations, which may be solved in parallel. The procedure is combined with finite element discretization in the spatial variables. The method is also applied to some parabolic type evolution equations with memory.

Key Words. Parabolic type, Laplace transform, parallel method and high order quadrature.

1. Introduction

In this paper we present a survey of recent work on an approach to time discetization of some equations of parabolic type based on Laplace transformation and quadrature. Following work by Sheen, Sloan, and Thomée [7], [8], we first introduce our method for an abstract parabolic equation, and then apply the method to the heat equation and its spatial discretization by finite elements, which produces a fully discrete scheme. We then describe work in McLean and Thomée [3] concerning application of the method to an evolution equation with a memory term of fractional integral type, and finally preview ongoing work by McLean, Sloan, and Thomée [5], where the method is used for a parabolic integro-differential equation with a memory term of convolution type. Our presentation here will be sketchy, and we refer to the original papers for details.

We consider the approximate solution of a parabolic problem of the form

$$u_t + Au = f(t),\ \text{for}\ t > 0, \quad \text{with}\ u(0) = u_0, \tag{1.1}$$

where u_0 and $f(t)$ are given. Having in mind the case that A is a second order elliptic differential operator with Dirichlet boundary conditions in a spatial domain Ω, we consider the problem in the framework of a Banach space $\mathbb{B}$. We assume that A is a closed operator in $\mathbb{B}$ such that $-A$ generates a bounded analytic semigroup $E(t) = e^{-At}$. More precisely, we assume that the spectrum $\sigma(A)$ of A is contained in a sector of the right half plane, and that that the resolvent $(z\,I + A)^{-1}$ of $-A$ satisfies

$$\|(z\,I + A)^{-1}\| \le M(1 + |z|)^{-1},\ \text{for}\ z \in \Sigma_\delta = \{z : |\arg z| < \delta\}, \tag{1.2}$$

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with $\delta \in (\pi/2, \pi)$ and M independent of z. When A is symmetric and positive definite in a Hilbert space, δ can be chosen as an arbitrary number in $(\pi/2, \pi)$, and $M = O((\pi - \delta)^{-1})$. Here we shall consider δ and M fixed. For the elliptic differential operator case and $\mathbb{B} = C_0(\bar{\Omega})$, (1.2) was shown in Stewart [9].

The first step in our approach is to represent the solution $u(t)$ as a contour integral of the form

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} w(z)\, dz, \tag{1.3}$$

where $w(z)$ is the Laplace transform of u,

$$w(z) = \widehat{u}(z) = \int_0^\infty e^{-zt} u(t)\, dt, \text{ for } Re\ z \geq x_0, \tag{1.4}$$

with $x_0 \in R$, and where initially Γ is an appropriately chosen line Γ_0 in the complex plane parallel to the imaginary axis. In (1.3), $u(t)$ is then just the inverse Laplace transform of $w(z)$. For our purposes, however, assuming that $w(z)$ may be continued analytically in an appropriate way, we shall want to take for Γ a deformed contour in the set Σ_δ in (1.2), which behaves asymptotically as a pair of straight lines in the left half plane, with slopes $\pm\sigma \neq 0$, say, so that the factor e^{zt} decays exponentially as $|z| \to \infty$ on Γ.

For concreteness, we take

$$\Gamma = \{z : z = \varphi(y) + i\sigma y,\ y \in R\} \subset \Sigma_\delta, \quad \varphi(y) = \gamma - \sqrt{y^2 + \nu^2}, \tag{1.5}$$

for suitable positive parameters γ, ν, and σ . The curve Γ is then the left-hand branch of a hyperbola, which crosses the real axis at $\alpha = \varphi(0) = \gamma - \nu$. Some of the constants below will depend on the parameters of Γ.

Taking Laplace transforms in (1.1), we obtain the transformed equation

$$(z\, I + A) w(z) = u_0 + \widehat{f}(z), \tag{1.6}$$

and thus $w(z)$ may be written formally as

$$w(z) = (z\, I + A)^{-1}(u_0 + \widehat{f}(z)), \text{ for } z \in \Gamma. \tag{1.7}$$

We assume that the Laplace transform $\widehat{f}(z)$ has an analytic continuation from Γ_0 to our deformed contour Γ, so that all singularities of $\widehat{f}(z)$ lie to the left of Γ. The same property will then apply to $w(z)$ in (1.7).

Using our assumptions on A, Γ, and $\hat{f}(z)$, one may use this representation of $u(t)$ to show the following stability and smoothness estimate.

Theorem 1.1. *We have for the solution $u(t)$ of (1.1), for $j, k \geq 0$,*

$$\|A^j u^{(k)}(t)\| \leq C t^{-k} e^{\alpha t} (\|u_0\| + \|\hat{f}\|_\Gamma), \text{ for } t > 0, \quad \text{where } \|\hat{f}\|_\Gamma = \sup_{z \in \Gamma} |\hat{f}(z)|.$$

In terms of the analytic semigroup $E(t)$ we have for the solution of (1.1)

$$u(t) = E(t) u_0 + \int_0^t E(t-s) f(s)\, ds, \text{ for } t \geq 0.$$

Since $\|E(t)\| \leq C_0$ for $t \geq 0$ and for some $C_0 \geq 1$, one has the stability property

$$\|u(t)\| \leq C_0 \big(\|u_0\| + \int_0^t \|f(s)\|\, ds\big), \text{ for } t \geq 0. \tag{1.8}$$