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A POSTERIORI ERROR ESTIMATOR FOR FINITE ELEMENT DISCRETIZATIONS OF QUASI-NEWTONIAN STOKES FLOWS

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Abstract. In this paper, we consider mixed finite elements discretizations of a class of Quasi-Newtonian Stokes flow problem. Unified a posteriori error estimator for conforming, nonconforming, with or without stabilization is obtained. We prove, without Helmholtz decomposition of the error, nor regularity and saturation assumptions, the reliability and the efficiency of our estimator.

Key Words. Quasi-Newtonian flow, conforming, nonconforming and mixed finite element, a posteriori error estimator.

1. Introduction

Adaptive finite element method is justified by using a posteriori error estimate which provides computable upper and lower error bounds, it serves then, as error indicators. The aim of the work is to unify, generalize and refine the derivation of residual error estimator for a class of Quasi-Newtonian Stokes flow problem. Indeed, the present work take on unifying proof for conforming, nonconforming, and even conforming-nonconforming scheme, with or without stabilization [4], and also mixed formulation, in two and three dimensional cases [9]. We generalize, simplify and refine the works of Verfürth [12] , Dari, Durán a nd Padra [8], Carstensen and Funcken [6] and Gatica et al [9]. We prove, without Helmholtz decomposition of the error, nor regularity of the solution or the domain, nor saturation assumption, the efficiency and the reliability of our estimator.
Let $\Omega \subset I\!R^d$ (d=2,3), be a bounded open connected and polyhedral set. In Ω, we consider the following model problem:

$$\left\{ \begin{array}{rl} \text{Find} & (u,p) \text{ such that} \\ -div(\mathcal{A}(\nabla u)) + \nabla p & = f, \;\text{ in } \Omega, \\ divu & = 0, \text{ in } \Omega, \\ u & = 0, \text{ on } \Gamma = \partial\Omega, \end{array} \right.$$

where u the velocity, p the pressure, f a regular function in the sapce $(L^2(\Omega))^d$ and $\mathcal{A} : I\!R^{d\times d} \longrightarrow I\!R^{d\times d}$ is Lipschitz continuous function satisfying, there are positives constants c_1 and c_2 such that: for all $\alpha, \beta \in I\!R^{d\times d}$,

$$c_1\|\alpha - \beta\|^2 \leq (\mathcal{A}(\alpha) - \mathcal{A}(\beta)) : (\alpha - \beta), \tag{1.1}$$

and

$$\|\mathcal{A}(\alpha) - \mathcal{A}(\beta)\| \leq c_2\|\alpha - \beta\|, \tag{1.2}$$

(Colon denotes the scalar product in $I\!R^{d\times d}$).

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This kind of nonlinear Stokes problem appears in the modeling of a large class of non-Newtonian fluids. In the particular case of Carreau law for viscoelastic flows (see, e.g. [11]), we have

$$\forall \alpha \in \mathbb{R}^{d\times d}, \quad \mathcal{A}(\alpha) = (k_0 + k_1(1+\|\alpha\|^2)^{\frac{\beta-2}{2}})\alpha,$$

with $k_0 \geq 0, k_1 > 0$ and $\beta \geq 1$. It is easy to verify that the Carreau law satisfies (1.1) and (1.2) for all $k_0 > 0$ and $\beta \in [1,2]$. In particular, with $\beta = 2$ we find the usual linear Stokes model.

In the sequel, we denote by $W^{s,p}(\Omega)$ and $W^{s,p}(\Gamma), 0 \leq s$ and $1 \leq p \leq +\infty$, the usual Sobolev spaces (see e.g [1]), endowed with the norms $||.||_{s,p,\Omega}$ and $||.||_{s,p,\Gamma}$ respectively. For a non integer s, we use the notations $|.|_{s,p,\Omega}$ and $|.|_{s,p,\Gamma}$, given explicitely, as following:

$$\text{if } p < +\infty, \qquad |v|^p_{s,p,\Omega} = \int\int_{\Omega\times\Omega} \frac{||D^{[s]}v(x) - D^{[s]}v(y)||^p}{|x-y|^{d+p\sigma}} dxdy,$$

$$\text{if } p = \infty, \qquad |v|_{s,+\infty,\Omega} = \sup_{\Omega\times\Omega} \frac{||D^{[s]}v(x) - D^{[s]}v(y)||^p}{|x-y|^{\sigma}}$$

and

$$|v|^p_{s,p,\Gamma} = \int\int_{\Gamma\times\Gamma} \frac{||D^{[s]}v(x) - D^{[s]}v(y)||^p}{|x-y|^{d-1+p\sigma}} dxdy,$$

where $[s]$ is the integer part of s and $\sigma = s - [s]$. $H^s(\Omega)$ is the usual space $W^{s,2}$ and $H^s_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$.

In order to state the precise form of our estimator, we specify the hypothesis on the class of finite elements spaces under questions. Let $\mathcal{T}_h$ be a family of regular triangulations by triangles or tetrahedron of Ω in the sens of Ciarlet [7], We denote by $\mathcal{N}$ the set of all nodes in $\mathcal{T}_h$, and by $\mathcal{K} := \mathcal{N}/\Gamma$ the set of free nodes. Let ϕ_a denotes a hat function for $a \in \mathcal{N}$ which is piecewise linear function such that $\forall b \in \mathcal{N} \;\; \phi_a(b) = \delta^b_a$, by $\omega_a := \{x \in \Omega, \phi_a(x) > 0\}$ we denote the patch of $a \in \mathcal{N}$ and we set $h_a := diam(\omega_a)$. Finally, we denoted by $\mathcal{E}$ the set of all edges (faces) of $\mathcal{T}_h$ and by $\mathcal{E}_I$ the set of all interior edges (faces) of $\mathcal{T}_h$.

We introduce the following spaces:

$$\begin{aligned} V_h = \;\; & \{v_h \in L^2(\Omega),\; \forall T \in \mathcal{T}_h,\; v_{h|T} \in (P_1(T))^d, \forall e \in \mathcal{E}_I, \;\; \int_e [v_h] d\sigma = 0, \\ & \forall e \text{ edge (face) } \subset \Gamma \qquad \int_e v_h d\sigma = 0\}, \end{aligned}$$

and

$$M_h = \{q_h \in L^2_0(\Omega), \forall T \in \mathcal{T}_h,\; q_{h|T} \in P_1(T)\},$$

In the sequel, we consider $(u_h, p_h) \in (V_h)^d \times M_h$ verifying: $\forall v_h \in (V_h \cap H^1_0(\Omega))^d$,

$$\sum_{T\in\mathcal{T}_h} \int_T \mathcal{A}(\nabla u_h).\nabla v_h - \sum_{T\in\mathcal{T}_h} \int_T p_h div v_h dx = \int_\Omega f.v_h dx, \tag{1.3}$$

For abbreviation, we frequently write $\|.\|_{1,h,\omega} = \{\sum_{T\in\mathcal{T}_h, T\subset\omega} \|.\|^2_{1,T}\}^{\frac{1}{2}}$ and neglect the domain when $\omega := \Omega$ if there is no risk of confusion, and we denote by div_h the operator defined from

$$H(div;\mathcal{T}_h) := \{\sigma \in (L^2(\Omega))^{d\times d}; \;\; \forall T \in \mathcal{T}_h, \sigma_{|T} \in H(div;T)\}$$

onto $L^2(\Omega)^d$ by:

$$\forall T \in \mathcal{T}_h, \quad div_h \sigma = div \sigma \qquad \text{on } T.$$