INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 3, Pages 241–282

MORTAR ADAPTIVITY IN MIXED METHODS FOR FLOW IN POROUS MEDIA

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Abstract. We define an error indicator for mixed mortar formulation of flow in porous media. The mixed mortar domain decomposition method for single-phase flow problems was defined by Arbogast et al; it relies on coupling of subdomain problems using mortar Lagrange multipliers defined as continuous piecewise linears on the subdomain interface. The accuracy and efficiency of the resulting interface formulation depends on the number of mortar degrees of freedom which we propose to adapt using error indicators involving jump of the flux across the interface. Rigorous a-posteriori analysis and proof of reliability of the estimator are established for single-phase 2D flow problems with diagonal coefficients for $\mathbf{RT}_{[0]}$ spaces on rectangular grids. Computational experiments demonstrate the application of the estimator. Next, the algorithm and indicator are extended to the two-phase flow case which is illustrated with numerical examples. We focus on adapting the mortar grid while keeping subdomain grids fixed. Full mortar adaptivity is discussed elsewhere.

Key Words. Single-phase flow in porous media, multi-phase flow, mixed finite elements, a-posteriori error estimation, mortars, domain decomposition, adaptivity

1. Introduction

This paper is devoted to grid adaptivity for a family of heterogeneous domain decomposition methods based on the *mixed mortar* finite element method.

The method was introduced in [7] and it provides a rigorous optimally convergent discretization technique for the elliptic equation

$$-\nabla \cdot (K\nabla p) = f, \quad x \in \Omega, \tag{1}$$

with $\Omega \subset \mathbb{R}^d, d = 2, 3$. Here K denotes the diffusion coefficient, f denotes the source/sink terms, and p is the unknown pressure.

In the mortar domain decomposition method, the region Ω is decomposed into individual non-overlapping subdomains $\Omega_i, i = 1, \dots n$ which are separated by the union of interfaces Γ on which *mortar* grids and unknowns are introduced. The subdomains are gridded independently; subdomain problems which are the local counterparts of (1) can be solved essentially independently from one another but are coupled by mortars; see [14, 12] for mortar formulation when subdomain problems are solved with standard Galerkin (conforming) methods.

In the mixed mortar method the subdomain problems are solved using mixed finite element methods thereby providing a locally conservative approximation to

Received by the editors April 15, 2004 and, in revised form, October 22, 2004.
2000 *Mathematics Subject Classification.* 65N30,65M60,65N50,65N55,76S05,76M10.
This research was partly supported by the DOE grant DE-FG03-99ER25371, and the NSF grants: SBR 9873326, ITR EIA-0121523, NPACI 10181410.

both pressure and velocity unknowns $\mathbf{u} := -K\nabla p$. The method relies on introducing *mortar Lagrange multipliers* on the interface Γ which provide Dirichlet boundary conditions for the subdomain problems. Additionally, the subdomain problems are coupled by the requirement that the global velocities be *weakly* continuous across Γ which relaxes the global continuity (of normal components across any smooth surface) of exact velocities. This weak-continuity condition averages the jumps of velocities and is defined relative to the discrete space of Lagrange multipliers on the interface which are defined on a *mortar grid* characterized by the parameter h_m, or by the number of mortar degrees of freedom $n_m \approx O(\frac{1}{h_m})$.

Let us be given a collection of rectangular partitions of Ω_i with associated grid parameter $h = \max_i h_i$. In principle, h_m can be selected independently of h, as long as certain lower and upper bound conditions hold. These guarantee, respectively, the unique solvability of the mixed mortar formulation, and the optimal approximation properties of weakly continuous velocities which in turn are necessary for the optimal rate of convergence of the method, the same as for discretization *without* mortars. For this optimal convergence rate which, for lowest order Raviart-Thomas spaces $\mathbf{RT}_{[0]}$, is $O(h)$ in both the pressure and velocity unknowns [46, 17], h_m should depend linearly on h; the approximation error increases, in general, with the proportionality constant.

The number of mortar unknowns n_m on Γ determines the complexity of the *interface problem*. Recall that, in a classical domain decomposition setting, the algorithm for approximation of (1) can be written in terms of the interface unknowns and as such solved by an iterative algorithm which requires, in each iteration, solution of subdomain problems which are responding to the current guess of Dirichlet data. In the mixed mortar algorithms the number of iterations on the interface in general grows with n_m, unless optimal preconditioners can be applied.

The mixed mortar method has been the cornerstone of several major reservoir simulation projects. Recall that (1) can be used as a model for single-phase flow in a reservoir Ω. Its natural extension is to the multi-phase flow; the algorithm has been integrated within the IPARS (Integrated Parallel Accurate Reservoir Simulator) framework [51, 44, 49, 41, 35]. The attractiveness of the mortar approach lies in that it makes the subdomain problems independent from one another. It is only the interface Lagrange multipliers (Dirichlet data) and the resulting fluxes (the Neumann data) which provide "communication" between subdomains. As such, the subdomain problems can be considered as "black-boxes" thereby allowing for local adaptivity of time-stepping [43], grids and solvers, and mainly, the physical models [37]. The latter coupling is a form of *multiphysics* and is an instance of *heterogeneous domain decomposition*; see [50].

The difficulties in practical application of the overall procedure lie in finding optimal preconditioners for the general multi-phase solver on the interface; see [57]. However, in spite of the large complexity of the interface solver, the mixed mortar approach has been extremely successful when applied to a large class of real reservoir problems. It is important to note that in all the successful cases we found a relatively small n_m sufficient for a good level of accuracy and at the same time mandatory for an acceptable degree of computational complexity.

In [45] the mixed mortar method gave rise to the *mortar upscaling* method. Here the subdomain grids are kept fixed but n_m varies, thereby providing a variable degree of local conservation of mass or of weak-continuity of the fluxes.

It is in the research reported in [43, 37, 45] that the need to define the "right" mortar grid and to control the error due to only weak continuity of fluxes became