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CONTROL OF GEOMETRY INDUCED ERROR IN *hp* FINITE ELEMENT(FE) SIMULATIONS
I. EVALUATION OF FE ERROR FOR CURVILINEAR GEOMETRIES

DONG XUE AND LESZEK DEMKOWICZ

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Abstract. The paper discusses a general framework for handling curvilinear geometries in high accuracy Finite Element (FE) simulations, for both elliptic and Maxwell problems. Based on the differential manifold concept, the domain is represented as a union of geometrical blocks prescribed with globally compatible, explicit or implicit parameterizations. The idea of parametric H^1-, $H(curl)-$ and $H(div)$ - conforming elements is reviewed, and the concepts of exact geometry elements and isoparametric elements are discussed. The paper focuses then on isoparametric elements, and two ways of computing FE discretization errors: a popular one, neglecting the geometry approximation, and a precise one, utilizing the exact geometry representation. Presented numerical examples indicate the necessity of accounting for the geometry error in FE error calculations., especially for the $H(curl)$ problems.

Key Words. Geometry approximation, curvilinear *hp* Finite Element (FE) meshes, error evaluation, Exact Geometry Integration (EGI).

1. Introduction

The *hp*-adaptive FE methods are some of the most powerful methodologies for simulating complex engineering problems. These numerical methods provide optimal sequences of *hp*-grids that achieve exponential convergence, whereas h or p method converges only, at best algebraically [1, 6]. The advantages of *hp* methods are achieved by the proper choice of meshing and mapping procedures to create a finite element mesh over an arbitrary domain.

Sizable errors are introduced into the prediction of parameters when the geometric approximation is too low w.r.t. [1] the polynomial order of the discretization. [15, 16] show the importance of using properly mesh entities in high order discretization to solve partial differential equations. Current development efforts in *hp* methods are aimed not only at a curvilinear mesh geometry representation over curved domains [12], but also at the effective definition of meshes consisting of mixed order elements.

Solving Boundary Value Problems(BVP) in complex geometries using *hp* finite elements consists of a double discretization. First, a mesh is introduced in order to

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[1] with respect to

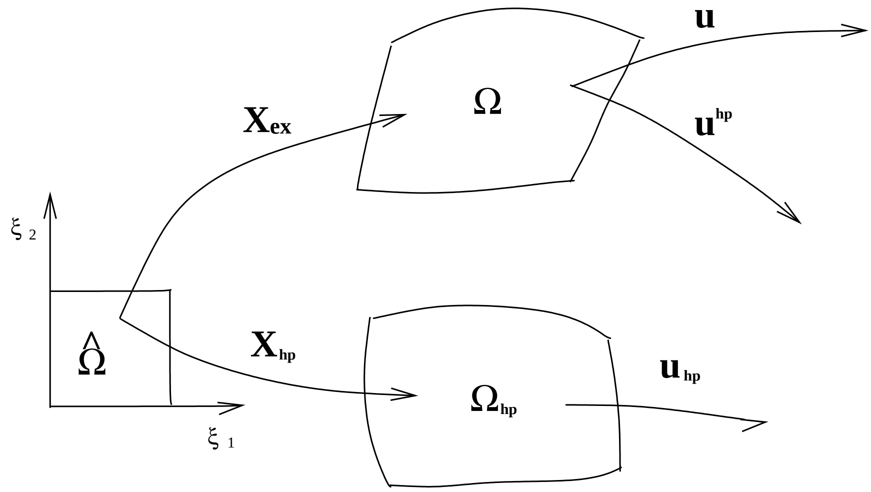

FIGURE 1. The exact and approximate domain of Finite Element Method.

create a discrete geometrical domain. Then, the solution function space is approximated by a finite dimensional function space. Both geometrical and function space approximations introduce discretization errors into the solution. The element level integral is represented abstractly as,

$$I = \int_{\Omega} \boldsymbol{K}(\boldsymbol{x}) d\boldsymbol{x} = \int_{\hat{\Omega}} \boldsymbol{K}(\boldsymbol{X}_{ex}(\boldsymbol{\xi})) d\boldsymbol{\xi}, \tag{1}$$

where K represents integrands associated with the interior of element domain Ω. The approximations can be introduced at one or more following basic functional levels: approximation of Ω, approximation of K, approximation of integration method over domain Ω. To evaluate the integral, the traditional method uses isoparametric geometry representation $\boldsymbol{X}_{hp}(\xi) \in \Omega_{hp}$, followed by the error integration on the approximate geometry domain Ω_{hp}. We will refer to it as the *Approximate Geometry Integration* (AGI),

$$\begin{aligned} I \approx I_{hp} &= \int_{\Omega_{hp}} \boldsymbol{K}_{hp}(\boldsymbol{x})\, d\,\boldsymbol{x} = \int_{\hat{\Omega}} \boldsymbol{K}_{hp}(\boldsymbol{X}_{hp}(\boldsymbol{\xi}))\, d\,\boldsymbol{\xi} \\ &\approx \sum_{\xi_l} \boldsymbol{K}_{hp}(\boldsymbol{X}_{hp}\,(\boldsymbol{\xi}_l))\omega_l. \end{aligned} \tag{2}$$

Here the weights $\boldsymbol{\omega}_l$ and quadrature points $\boldsymbol{\xi}_l$ are determined by the order of integration. Approximate geometry representation leads to inexact representation of boundary and initial conditions and, therefore, inappropriate evaluation of element level integrals. The exact solution $u :\ \Omega \to \mathbb{R}$ cannot be compared directly to the approximate solution $u_{hp} :\ \Omega_{hp} \to \mathbb{R}$ because they are computed on different physical domains, see Fig.1. This prompts us to develop a element mapping scheme resulting in a modified meaning of the FE solution defined on the exact physical domain:

$$u^{hp} :\ \Omega \to \mathbb{R} \tag{3}$$

Our study is primarily motivated with geometry induced error control. In this paper, we consider the following two issues:

- A proper definition of the geometry error and its assessment.