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L^∞-ERROR ESTIMATES AND SUPERCONVERGENCE IN MAXIMUM NORM OF MIXED FINITE ELEMENT METHODS FOR NONFICKIAN FLOWS IN POROUS MEDIA

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Abstract. On the basis of the estimates for the regularized Green's functions with memory terms, optimal order L^∞-error estimates are established for the nonFickian flow of fluid in porous media by means of a mixed Ritz-Volterra projection. Moreover, local L^∞-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived for the mixed finite element method, and global L^∞-superconvergence estimates for the velocity and the pressure are also investigated by virtue of an interpolation post-processing technique. Meanwhile, some useful a-posteriori error estimators are presented for this mixed finite element method.

Key Words. NonFickian flow, mixed finite element methods, the mixed Ritz-Volterra projection, Green's functions, error estimates and superconvergence

1. Introduction

The nonFickian flow of fluid in porous media can be modelled by an integro-differential equation which seeks $u = u(x,t)$ such that

$$
\begin{aligned}
u_t &= \nabla\cdot\sigma + cu + f && \text{in } \Omega\times J,\\
\sigma &= A(t)\cdot\nabla u - \int_0^t B(t,s)\cdot\nabla u(s)ds && \text{in } \Omega\times J,\\
u &= g && \text{on } \partial\Omega\times J,\\
u &= u_0(x) && x\in\Omega,\ t=0,
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset R^d$ $(d = 2,3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $J = (0,T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two 2×2 or 3×3 matrices, and A is positive definite, $c \le 0$, f, g and u_0 are known smooth functions. This kind of flow is complicated by the history effect characterizing various mixing length growth of the flow, which has been investigated, for example, in [9, 10] and references cited therein.

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The numerical approximations of the problem (1.1) are available in extensive literature. See, for instance, [2, 3, 12, 13, 16, 14, 15, 20, 21, 22], where some optimal order error estimates and superconvergence have been established.

In the present paper, the solutions of (1.1) are approximated by mixed finite element methods [14, 15, 16]. Optimal order L^∞-error estimates are obtained by employing a mixed Ritz-Volterra projection introduced in [16]. In addition, local L^∞-superconvergence estimates for the velocity along the Gauss lines and for the pressure at the Gauss points are derived, and with the aid of an interpolation post-processing method global L^∞-superconvergence estimates are also derived for the velocity and the pressure approximations. As a result of the global superconvergence, a-posteriori error indicators of the mixed finite element method are presented in the paper.

Compared with [16], where the optimal and superconvergence estimates of the mixed finite element method in L^2-norm have been discussed for the problem (1.1), the key point of the present paper is the introduction of the regularized Green's functions with memory terms and the establishment of the various estimates for them and their mixed finite element approximations, which will play an important role in the forthcoming analysis in deriving the above optimal and superconvergence L^∞-error estimates. As a result, the methodology and the techniques used in this paper are quite different from those in [16].

The paper is organized in the following manner. In Section 2, we give the approximate sub-space and the approximate problem. Two regularized Green's functions and a Ritz-Volterra projection with memory terms for the mixed form for the problem (1.1) are introduced in Section 3. Also, in Section 3 the L^1-error estimates and related estimates for the mixed finite element approximations of the regularized Green's functions are stated, and the L^∞-error estimates for the mixed Ritz-Volterra projection are established. In Section 4, optimal order error estimates in maximum norm are given for the mixed finite element approximations. Section 5 is devoted to the local and global L^∞-superconvergence analysis of the mixed finite element method, by which some a-posteriori error estimators are obtained for the mixed finite element method. Finally, the L^1-error estimates and related estimates for the mixed finite element approximations of the regularized Green's functions are proved in Section 6.

2. The mixed finite element method

In this section, we give the mixed finite element approximate scheme for the parabolic integro-differential equation (1.1). For simplicity, the method will be presented on plane domains.

Let $W := L^2(\Omega)$ be the standard L^2 space on Ω with norm $\|\cdot\|_0$. Denote by

$$\mathbf{V} := H(\mathrm{div}, \Omega) = \left\{\sigma \in (L^2(\Omega))^2 \mid \nabla\cdot\sigma \in L^2(\Omega)\right\}$$

the Hilbert space equipped with the following norm:

$$\|\sigma\|_{\mathbf{V}} := \left(\|\sigma\|_0^2 + \|\nabla\cdot\sigma\|_0^2\right)^{\frac{1}{2}}.$$

There are several ways to discretize the problem (1.1) based on the variables σ and u; each method corresponds to a particular variational form of (1.1) [14, 22].

Let T_h be a finite element partition of Ω into triangles or quadrilaterals which is quasi-uniform. Let $\mathbf{V}_h \times W_h$ denote a pair of finite element spaces satisfying