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NONSTANDARD NONCONFORMING APPROXIMATION OF THE STOKES PROBLEM, I: PERIODIC BOUNDARY CONDITIONS

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Abstract. This paper analyzes a nonstandard form of the Stokes problem where the mass conservation equation is expressed in the form of a Poisson equation for the pressure. This problem is shown to be wellposed in the d-dimensional torus. A nonconforming approximation is proposed and, contrary to what happens when using the standard saddle-point formulation, the proposed setting is shown to yield optimal convergence for every pairs of approximation spaces.

Key Words. Stokes equations, finite elements, nonconforming approximation, incompressible flows and Poisson equation

1. Introduction

Consider the Stokes equations in a bounded domain Ω:

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \nabla\cdot u = 0. \tag{1.1}$$

The objective of the present work is to analyze the following nonstandard form of the Stokes equations:

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \Delta p = \nabla\cdot f; \quad \partial_n p|_{\partial\Omega} = (-\nabla\times\nabla\times u + f)\cdot n|_{\partial\Omega}. \tag{1.2}$$

The Poisson equation for the pressure is obtained formally by taking the divergence of the momentum equation, and the Neumann boundary condition is obtained by taking the normal component of the momentum equation at the boundary of the domain and substituting $-\Delta u$ by $\nabla\times\nabla\times u$ since $\nabla\cdot u$ is expected to be zero (recall that $-\Delta u = -\nabla\nabla\cdot u + \nabla\times\nabla\times u$). This way of solving the Stokes (or Navier–Stokes) equations seems to be standard in the literature dedicated to the analysis of turbulence in the d-torus. It currently seems also to attract a growing interest in the literature dealing with the approximation of the time-dependent Stokes (and Navier–Stokes) equations. This form of the Stokes equations is one building block of a splitting algorithm proposed by Orszag et al. [7] and Karniadakis et al. [6]. This problem has also been shown to play an important role in a new type of splitting algorithm proposed in [5]. A recurrent claim in the literature about this strange

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form of the Stokes equations is that when discretized it does not require the velocity and the pressure spaces to satisfy the so-called Babuška–Brezzi condition, i.e., there are no spurious pressure modes. To the present time, this claim has never been proved. The main reason for the lack of proof is that, mathematically speaking, the problem (1.2) is far from being standard. Actually, this form of the problem is more prone to raise eyebrows of mathematically minded readers than to attract their interest. Since the usual setting for this problem is to assume that f is in $[H^{-1}(\Omega)]^d$, the velocity is in $[H^1(\Omega)]^d$ and the pressure is in $L^2(\Omega)$. This type of regularity is incompatible with the boundary condition $\partial_n p|_{\partial\Omega} = (\nabla\times\nabla\times u + f)\cdot n|_{\partial\Omega}$ since it is not legitimate to speak of the normal derivative of a function in $L^2(\Omega)$, nor is it legitimate to speak of the normal component of a $\mathbb{R}^d$-valued distribution in $[H^{-1}(\Omega)]^d$.

This work is an attempt at tackling the above issue. We first analyze a slightly modified version of (1.2), (See problem (2.1)) and we show that this modified version is wellposed. Although, we do not solve exactly (1.2), we think that the setting used for the analysis of the modified problem gives hints of what should be used to seriously tackle (1.2). Since the bothering issue in (1.2) is the boundary condition, in the second part of this work we analyze (1.2) in the periodic d-torus. To the best of our knowledge, the analysis of this problem using finite elements does not seem to have been done yet. In this setting we are able to conduct a full analysis. We propose a discrete formulation and we show that it is optimally convergent. The main result of the paper is Theorem 3.1. The main conclusion of our analysis is that, yes indeed, (1.2) in the d-torus yields an optimal approximation setting that does not require the approximation spaces to satisfy the Babuška–Brezzi condition.

2. The continuous problem

This section is composed of two subsections. First we consider a slightly modified version of (1.2), which we prove to be wellposed. Second we analyze (1.2) adopting periodic boundary conditions.

2.1. First formulation. The problem that we consider can be written formally in the following form

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \Delta\nabla\cdot u = 0; \quad \partial_n\nabla\cdot u|_{\partial\Omega} = 0. \tag{2.1}$$

To give sense to the above problem we introduce the spaces

$$X = [H^1_0(\Omega)]^d; \quad M = L^2_{\int=0}(\Omega); \quad Z = \{\phi \in H^1_{\int=0}; \Delta\phi \in L^2(\Omega); \partial_n\phi|_{\partial\Omega} = 0\}, \tag{2.2}$$

where $L^2_{\int=0}(\Omega)$ is composed of those functions in $L^2(\Omega)$ whose mean-value is zero. We equip X, M and Z with the following norms $\|u\|_X = \|u\|_{1,\Omega}$, $\|p\|_M = \|p\|_{0,\Omega}$, $\|q\|_Z = \|q\|_{1,\Omega} + \|\Delta q\|_{0,\Omega}$, where $\|\cdot\|_{s,\Omega}$ denotes the norm in $H^s(\Omega)$. No notational distinction is made between the norm of scalar-valued and vector-valued functions. The product spaces $X\times M$ and $X\times Z$ are equipped with the norms $\|(u,p)\|_{X\times M} = \|u\|_X + \|p\|_M$ and $\|(u,p)\|_{X\times Z} = \|u\|_X + \|p\|_Z$. All the above normed vector spaces