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# NUMERICAL APPROXIMATION OF TWO-DIMENSIONAL CONVECTION-DIFFUSION EQUATIONS WITH MULTIPLE BOUNDARY LAYERS

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**Abstract.** In this article, we demonstrate how one can improve the numerical solution of singularly perturbed problems involving multiple boundary layers by using a combination of analytic and numerical tools. Incorporating the structures of boundary layers into finite element spaces can improve the accuracy of approximate solutions and result in significant simplifications. We discuss here convection-diffusion equations in the case where both ordinary and parabolic boundary layers are present.

**Key Words.** boundary layers, finite elements, singularly perturbed problem, convection-diffusion

## Contents



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## 1. Introduction

In this article we consider linear singularly perturbed convection dominated boundary value problems of the following types:

$$L_\epsilon u^\epsilon := -\epsilon \triangle u^\epsilon - u^\epsilon_x = f(x,y) \quad \text{for } (x,y) \in \Omega, \tag{1.1a}$$

with boundary conditions

$$u^\epsilon = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

or,

$$\begin{aligned} u^\epsilon &= 0 \quad \text{at } x = 0, 1, \\ \frac{\partial u^\epsilon}{\partial y} &= 0 \quad \text{at } y = 0, 1. \end{aligned} \tag{1.1c}$$

Here $0 < \epsilon << 1$, and $\Omega = (0,1) \times (0,1) \subset \mathbf{R}^2$.

It can be shown (see below) that $u^\epsilon \to u^0$ in $L^2$ where $u^0$ is the solution of the limit problem:

$$-u^0_x = f \quad \text{in } \Omega, \tag{1.2a}$$

$$u^0 = 0 \quad \text{at } x = 1, \tag{1.2b}$$

so that we have

$$u^0 = \int_x^1 f(s,y)ds. \tag{1.2c}$$

Comparison between $u^\epsilon$ and $u^0$ is not easy because many discrepancies between $u^\epsilon$ and $u^0$ appear at the boundary. Just proving the $L^2$- convergence of $u^\epsilon$ to $u^0$ (which is a byproduct of the analysis below) is not straightforward. For a comparison between $u^\epsilon$ and $u^0$ in smaller spaces (spaces of more regular functions), we need to introduce a number of boundary layers of different types to account for the discrepancies. The most common boundary layer appears at $x = 0$ since $u^0(0, y)$ does not vanish in general; this boundary layer is obtained using the technique of ordinary boundary layers (OBL). From (1.2c), we see also that some discrepancies appear in general at the boundaries $y = 0, 1$. These will be accounted for by a less common concept of boundary layer, namely the parabolic boundary layer (PBL).

In [11] we discussed the problem (1.1a), (1.1b) when $f(x,y) = f_{yy}(x,y) = 0$ at $y = 0, 1$. In this case we only observe the discrepancy at $x = 0$ (note that $u^0(x, 0) = u^0(x, 1) = 0$), and the problem was thus handled by an OBL. In [16] we discussed equation (1.1a) in a channel with (1.1b) at $y = 0, 1$ and periodicity in the $x$- direction; in this case we only observe parabolic boundary layers (PBL).

Here, by considering equation (1.1a) in a square, we theoretically and numerically investigate the case where both OBLs and PBLs are present. In fact some restriction (compatibility conditions) will be assumed on $f$; indeed, as shown in [26], in the most general case (square with no restriction on $f$), several other inconsistencies occur which have to be accounted for by still other boundary layers. In this article, as we said, we avoid these additional boundary layers, and consider cases where only OBLs and PBLs are present. In fact we will see that for the mixed boundary value problem (1.1a), (1.1c), the compatibility conditions on $f$ and the effects of the PBLs are mild (see Section 4), whereas for (1.1a), (1.1b) we fully show how to overcome this compatibility condition issue.

Through the boundary layer analysis in Section 2, we will find rigorously that OBLs occur at the outflow $x = 0$ and PBLs occur at the characteristic lines $y = 0, 1$.