INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 4, Pages 409–421

BASIS FOR THE QUADRATIC NONCONFORMING TRIANGULAR ELEMENT OF FORTIN AND SOULIE

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Abstract. A basis for the quadratic (P_2) nonconforming element of Fortin and Soulie on triangles is introduced. The local and global interpolation operators are defined. Error estimates of optimal order are derived in both broken energy and $L^2(\Omega)$-norms for second-order elliptic problems. Brief numerical results are also shown.

Key Words. Quadratic nonconforming element, finite element method, error analysis

1. Introduction

Recently the nonconforming finite element method draws increasing attention from scientists and engineers as it has successfully provided stable numerical solutions to many practical fluid and solid mechanics problems: see, for instance, [1, 8, 9, 12, 15, 16, 17, 18, 28, 29, 32] for linear or nonlinear Navier-Stokes problems and [2, 5, 10, 13, 14, 21, 22, 19, 24, 23, 26, 27, 33] for elasticity related problems, and the references therein.

In order to approximate the velocity and pressure by the finite element method based on triangulations, the use of the usual P_1-P_0 conforming finite element pair lacks in stability that is required to satisfy the discrete inf-sup condition [7]. Also the P_1 conforming element suffers from numerical locking when applied to approximate elasticity problems [3, 6]. In case the triangulation is based on quadrilaterals rather than on triangles and the Q_1 element is used instead of the P_1 element accordingly, similar instability patterns are inevitable.

A common and simple solution to resolve this kind of instability problems has been made by using nonconforming finite element instead. In 1973 Crouzeix and Raviart [12] introduced the linear nonconforming finite elements for triangles or tetrahedrons and a cubic nonconforming element for triangles. The idea, at least in the P_1 nonconforming element case, is to employ the degrees of freedom associated with the values at the midpoints of edges of triangles or those at the centroids of faces of tetrahedrons, by replacing the values imposed at the vertices in the conforming element cases. These nonconforming elements were shown to supply stable finite element pairs for Stokes problems and to give optimal orders of convergence [12].

A generalization of this idea to higher degree nonconforming elements requires the patch test [20], which implies that a P_k nonconforming element needs to satisfy that on each interface the jump of adjacent polynomials be orthogonal to P_{k-1}

Received by the editors July 30, 2005.

2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

This research was supported in part by KRF 2003-070-C00007 and KOSEF R14-2003-019-01000-0.

polynomials on the interface. This implies that a P_2 nonconforming element must be continuous at the two Gauss points on each edge. However, to define the degrees of freedom at the two Gauss points leads to a problem due to the existence of a quadratic polynomial that vanishes at the six Gauss points of edges of any triangle. Therefore, the definition of the degrees of freedom for P_2 nonconforming elements addresses a special attention.

A successful quadratic nonconforming element has been introduced by Fortin and Soulie [17] by adding nonconforming bubble functions, "semi-loop functions", which can be eliminated by static condensation. (The three dimensional analogue has been introduced by Fortin [16].) It is shown in [17] that the global P_2 nonconforming space is identical to the union of the standard quadratic conforming element space, say, V_h, and the space of semi-loop functions, say Φ_h; moreover, their intersection $V_h \cap \Phi_h$, that is, "the set of the globally conforming and locally P_2 bubble functions", turns out to be one dimensional space. Therefore, in implementing the P_2 nonconforming element, one can modify the P_2 conforming code with a suitable addition of nonconforming bubble functions.

The purpose of the present paper is to propose another set of global basis functions for the P_2 nonconforming element, which have more nonconforming structure. For this, we define three kinds of basis functions with local supports: (1) edge-based basis (which are nonconforming), (2) vertex-based basis (which are nonconforming), and (3) triangle-based basis functions (which are bubble functions that are nonconforming). We then show that these functions form a basis for the P_2 nonconforming element space. The associated degrees of freedom and an interpolation operator are also defined. The basis functions corresponding to rectangular elements are constructed in [25].

The plan of the paper is as follows. In §2 the P_2 nonconforming basis functions are defined on triangular meshes. Interpolation and projection operators are defined in §3 and optimal order error estimates are shown. Finally in §4, brief numerical results are shown.

2. The P_2-nonconforming element on triangular meshes

In this section we introduce three kinds of basis functions for the P_2-nonconforming finite element on triangular meshes. The dimensions and basis functions are then computed for both Dirichlet and Neumann problems.

2.1. The P_2-nonconforming triangular elements. For a triangle T with the vertices $v_j, 1 \leq j \leq 3$, denote by $e_j, 1 \leq j \leq 3$, the edges from v_{j+1} to v_{j+2}, respectively, with the identification $v_1 = v_4$ and $v_2 = v_5$. Also, let m_j be the midpoint of $e_j, j = 1, 2, 3$. Throughout the paper we shall assume that the vertex indices are oriented counter clockwise. Designate by τ the unit tangent vector on the boundary ∂T with the direction from v_{j+1} to v_{j+2}, respectively, and by $\frac{\partial\varphi}{\partial\tau}$ its tangential derivative. Let g be the barycenter of T. As usual, for a nonnegative integer k, denote by $P_k(T)$ and $P_k(e_j)$ the spaces of polynomials on T and e_j, respectively, of degree $\leq k$. We begin with the following fact.

Lemma 2.1. *Let e be an edge of T. Then if $\varphi \in P_2(T)$ satisfies $\int_e \varphi\, ds = 0$ and $\int_e \frac{\partial\varphi}{\partial\tau}\, ds = 0$, φ vanishes at the two Gauss points on e.*

Proof. Since $\frac{\partial\varphi}{\partial\tau} \in P_1(e)$, $\int_e \frac{\partial\varphi}{\partial\tau}\, ds = |e|\, \frac{\partial\varphi}{\partial\tau}(m) = 0$, where $|e|$ denotes the length of e. This implies that $\varphi|_e$ is symmetric with respect to the midpoint of e. Then $\int_e \varphi\, ds = 0$ implies that $\varphi|_e \in P_2(e)$ vanishes at the two Gauss points on e. This proves the lemma. $\square$