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NUMERICAL SOLUTIONS TO BEAN'S CRITICAL-STATE MODEL FOR TYPE-II SUPERCONDUCTORS

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Abstract. In this paper we study the numerical solution for an $p-$Laplacian type of evolution system $\mathbf{H}_t + \nabla \times \left[|\nabla \times \mathbf{H}|^{p-2} \nabla \times \mathbf{H} \right] = \mathbf{F}(x,t), p > 2$ in two space dimensions. For large p this system is an approximation of Bean's critical-state model for type-II superconductors. By introducing suitable transformation, the system is equivalent to a nonlinear parabolic equation. For the nonlinear parabolic problem we obtain the numerical solution by combining approximation schemes for the linear equation and the nonlinear semigroup. The convergence and stability of the scheme are proved. Finally, a numerical experiment is presented.

Key Words. Approximation of Bean's Critical-State model, Numerical solutions.

1. Introduction

Bean's critical-state model for type-II superconductors describes the evolution of a magnetic field in an alloy-type of metal material under the external force ([4, 7]). The electric field $\mathbf{E}$ and the current density $\mathbf{J} = \sigma \mathbf{E}$ by Ohm's law are characterized as follows: there exists a critical current, denoted by J_c, such that $|\mathbf{J}| \leq J_c$ and

$$|\mathbf{E}| = \begin{cases} 0, & \text{if } |\mathbf{J}| < J_c, \\ [0, \infty), & \text{if } |\mathbf{J}| = J_c, \\ \emptyset, & \text{if } |\mathbf{J}| > J_c. \end{cases}$$

Here and thereafter, a bold letter represents a vector or vector function in R^3.

If one scales the value of critical current by assuming $J_c = 1$ without loss of generality, then the graph of $|\mathbf{E}|$ and $|\mathbf{J}|$ can be obtained formally from Ampere's law:

$$\mathbf{E} = |\nabla \times \mathbf{H}|^{p-2} \nabla \times \mathbf{H}$$

as $p \to \infty$, where $\mathbf{H}$ represents the magnetic field and the resistivity $\rho = \frac{1}{\sigma}$ is equal to $|\nabla \times \mathbf{H}|^{p-2}$.

This leads us to consider the following problem:

$$\mathbf{H}_t + \nabla \times \left[|\nabla \times \mathbf{H}|^{p-2} \nabla \times \mathbf{H} \right] = \mathbf{F}(x,t), \qquad (x,t) \in Q_T, \tag{1.1}$$

$$\nabla \cdot \mathbf{H} = 0, \qquad (x,t) \in Q_T, \tag{1.2}$$

$$\mathbf{n} \times \mathbf{H}(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1.3}$$

$$\mathbf{H}(x,0) = \mathbf{H}_0(x), \qquad x \in \Omega, \tag{1.4}$$

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where Ω is a bounded simply-connected domain in R^3 and $Q_T = \Omega \times (0, T]$, $p \geq 2$ is fixed, $\mathbf{n}$ is the outward unit normal on $\partial\Omega$ and $x = (x_1, x_2, x_3)$, $\mathbf{F}(x, t)$ represents the applied magnetic current.

For large p the electric resistivity $\rho = |\nabla \times \mathbf{H}|^{p-2}$ is small in the region $S_\varepsilon = \{(x, t) : |\nabla \times \mathbf{H}| \leq 1 - \varepsilon\}$ while it is very large in $\{(x, t) : |\nabla \times \mathbf{H}| \geq 1 + \varepsilon\}$, where ε is a small constant. Thus, the resistivity ρ in S_ε becomes smaller and smaller as p increases and eventually S_ε becomes the superconductor region as $\varepsilon \to 0$ (no resistivity). The region $\{(x, t) : 1 - \varepsilon < |\nabla \times \mathbf{H}| < 1 + \varepsilon\}$ is the transition zone and formally becomes a sharp interface between the normal and superconductor regions as $\varepsilon \to 0$.

Unlike Ginzburg-Landau's model for superconductors (see [7, 8]), the model problem (1.1)-(1.4) describes the macro-motion of magnetic currents and is often used by experimental physicists in searching of Type-II superconductor materials. For two-space dimensions, by a variational argument Prigozhin [13] proved the existence of a unique weak solution to Bean's model. Well-posedness of the problem (1.1)-(1.4) were established in [17] for R^3 and for a bounded simply-connected domain in R^3 in [18]. Regularity of the weak solution as well as the limit solution as $p \to \infty$ were also investigated in these papers (see [17, 18]). Particularly, the authors of [18] established the existence of a unique weak solution to Bean's model in three-space dimensions. Several researchers have investigated numerical solutions for Bean's model. Bossavit [1] and Prigozhin [14] studied the numerical solutions for the case where $\mathbf{H}$ has only one non-zero component. More recently, the authors of [10] studied the numerical solution of Bean's model via a finite element method and derived error estimate. For the problem (1.1)-(1.4), Barrett and Prigozhin in [3] discussed the finite element solution by assuming that $\mathbf{H}$ has one non-zero component. In the present paper, we study the numerical solution for the problem (1.1)-(1.4) in two space dimensions. By using a suitable transformation, we convert the system (1.1) to a nonlinear parabolic equation with possible degeneracy in the leading term. Based on the nonlinear semigroup theory, an algorithm is presented by a finite difference method. We calculate the numerical solution by solving a linear heat equation and using time-marching iteration technique. The method is quite easy to implement. We also show that this numerical method is convergence and stable in L^1 if we choose parameter properly. Moreover, the numerical scheme is unconditionally stable. Finally, a numerical experiment is given to verify our result.

The paper is organized as follows. In § 2, we transform the problem into a fully nonlinear parabolic equation. In § 3, we first present a numerical method to solve the nonlinear parabolic problem and then prove the convergence and stability of the numerical solution. In § 4 a numerical experiment is presented.

2. A New Formulation of the Problem

Consider the problem (1.1)-(1.4) in two space dimensions. Assume that $\mathbf{H}$ and $\mathbf{F}$ depend on (x_1, x_2) and component in z-direction is zero, i. e., $\mathbf{H}(x, t) = \{h_1(x, t), h_2(x, t), 0\}$, $\mathbf{F} = \{f_1(x, t), f_2(x, t), 0\}$,$\mathbf{H}_0 \in C^{2+\alpha}(R^2)$, $\mathbf{F} \in C^{2+\alpha}(0, T; R^2)$, $\mathbf{H}_0$ and $\mathbf{F}$ have compact support.

By the Theorem 2.2 of [17], we know that the problem (1.1)-(1.4) has a unique weak solution $\mathbf{H}(x, t)$ in $Q_T = \mathbf{R}^2 \times (0, T]$. Moreover $\mathbf{H}_t \in L^2(\mathbf{R}^2 \times (0, T])$, $\mathbf{H} \in L^\infty(0, T; B^d(\mathbf{R}^2))$, where $B^d(\mathbf{R}^2) = \{\mathbf{G}(x) \in W^{1,p}(\mathbf{R}^2) : \nabla \cdot \mathbf{G} = 0$, a. e. $x \in \mathbf{R}^2\}$. Moreover, $\mathbf{H}(x, t)$ has compact support for each $t \in [0, T]$. Therefore we can assume that Ω is sufficiently large such that the weak solution $\mathbf{H}$ satisfies