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MULTI-ALGORITHMIC METHODS FOR COUPLED HYPERBOLIC-PARABOLIC PROBLEMS

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Abstract. We study computational methods for linear, degenerate advection-diffusion equations leading to coupled hyperbolic-parabolic problems. A multi-algorithmic approach is proposed in which a different approximation method is used locally depending on the mathematical nature of the problem. Our analysis focuses on stability and a priori error estimates of coupled continuous and discontinuous Galerkin methods, achieving a global $h^{p+\frac{1}{2}}$ estimate. Both the mathematical analysis and the numerical results demonstrate that careful consideration is necessary when defining appropriate interface conditions between the hyperbolic and parabolic regions.

Key Words. discontinuous Galerkin, NIPG, interface conditions, porous media, coupled hyperbolic/parabolic PDE's.

1. Introduction

This work is motivated by the study of flow and transport phenomena in highly heterogeneous porous media, an important application in the petroleum and environmental industries. An appealing technique for handling such phenomena is the use of a multi-algorithmic strategy based on the decomposition of the spatial domain into multiple non-overlapping subdomains according to the geological, physical and chemical properties of the medium. This promotes the use of a different scheme within each subdomain in order to reduce computational expenses while preserving accuracy. The resulting numerical models are consistent with the underlying equations on the subdomains and physically meaningful conditions are imposed on interfaces between the subdomains. Examples of such domain decomposition approaches include the mortar finite element method employing Lagrange multipliers to weakly impose flux-matching across interfaces [1, 2] and related multi-block multi-physics techniques [3, 4].

We consider the specific case of advective-diffusive transport of a chemical species within strongly contrasting geological layers where the resulting diffusion coefficient varies spatially. As a model problem, we investigate advection-diffusion equations where diffusion is locally degenerate within the computational domain, leading to a coupled hyperbolic-parabolic problem. This situation lends itself to the use of domain-decomposition type coupled continuous Galerkin (CG) and discontinuous Galerkin (DG) methods where the strengths of each method are exploited within

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the appropriate subdomain. DG methods possess several characteristics which render them useful in many applications. Some well known versions include the local discontinuous Galerkin method of Cockburn and Shu [5], the OBB method of Oden, Babuška and Baumann [6], and the non-symmetric interior penalty Galerkin (NIPG) method of Rivière, Wheeler and Girault [7]. DG methods can efficiently handle advection-dominated flows since they generally exhibit less numerical diffusion than traditional CG methods, which efficiently handle diffusion-dominated flows. Furthermore the flexibility of DG methods allows for varying polynomial degree approximation and general non-conforming meshes. However, DG methods are more computationally expensive than CG methods since the degrees of freedom are associated with the elements rather than with the nodes. Thus, the use of a DG method on the hyperbolic subdomain coupled to a CG method on the parabolic region seems a natural choice.

When dealing with coupled hyperbolic-parabolic problems, special consideration must be paid to the interface conditions between parabolic and hyperbolic regions. For linear hyperbolic-parabolic problems such as those considered in this work, the interface conditions are relatively well understood. Gastaldi and Quarteroni [8] use a vanishing viscosity singular perturbation analysis to derive interface conditions for coupled hyperbolic-parabolic problems, which we employ henceforth. While the normal component of the total advective-diffusive flux is always continuous across the interface to ensure mass conservation, this is not the case for the solution itself. The latter is indeed continuous only at the subset of the interface where the flow leaves the parabolic subdomain and enters the hyperbolic region. On the other part of the interface, the solution is in general discontinuous. Recently, Croisille et al. [9] have used these interface conditions in the framework of the evolution linear semi-groups theory to establish a well-posedness result for one-dimensional, periodic, degenerate advection-diffusion equations.

In the case of a non-degenerate spatially-dependent diffusion coefficient yielding both advection-dominated and diffusion-dominated subdomains, it may be more useful to employ a DG method everywhere in the domain. A cost-effective strategy is to use, for instance, an NIPG method in the diffusion-dominated subdomain and a DG method in the advection-dominated subdomain. However, it remains important to be aware of the theoretical interface conditions when the Peclet number is sufficiently small to locally impose hyperbolic-type behavior in the solution. To pinpoint the main mathematical issues in this situation, we analyze a coupled NIPG/DG method for hyperbolic/parabolic problems.

This paper is organized as follows. The subsequent section defines the hyperbolic-parabolic problem and interface conditions under consideration. Section three analyzes a coupled CG/DG method and presents some convergence results as well as numerical results obtained with the coupled scheme on a model problem. Motivated by this approach and the careful treatment of terms in the resulting discretizations arising on the interface, section four then considers a scheme coupling the NIPG method in the parabolic subdomain with the DG method in the hyperbolic subdomain. The coupled NIPG/DG scheme is analyzed and convergence results are presented. In particular, we emphasize the fact that special consideration should be devoted to the design of interface conditions. The numerical results demonstrate that this difficulty must be tackled even in fully parabolic problems at the interface of those subdomains where the diffusion is dominated by the advection term, a difficult problem reflective of the small Peclet number associated with the equation. Conclusions are drawn in section five.