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# WAVEFORM RELAXATION METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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**Abstract.** $L^p$-convergence of waveform relaxation methods (WRMs) for numerical solving of systems of ordinary stochastic differential equations (SDEs) is studied. For this purpose, we convert the problem to an operator equation $X = \mathbb{T}X + G$ in a Banach space $\mathcal{E}$ of $\mathcal{F}_t$-adapted random elements describing the initial- or boundary value problem related to SDEs with weakly coupled, Lipschitz-continuous subsystems. The main convergence result of WRMs for SDEs depends on the spectral radius of a matrix associated to a decomposition of $\mathbb{T}$. A generalization to one-sided Lipschitz continuous coefficients and a discussion on the example of singularly perturbed SDEs complete this paper.

**Key Words.** waveform relaxation methods, stochastic differential equations, stochastic-numerical methods, iteration methods, large scale systems

## 1. Introduction

The solution of complex and large scale systems plays a crucial role in recent scientific computations. In particular, large scale stochastic dynamical systems represent very complex systems incorporating the random appearances of physical processes in nature. The development of efficient numerical methods to study such large scale systems, which can be characterized as weakly coupled subsystems with quite different behavior, is an important challenge. Under some conditions, block-iterative methods are very efficient. One of these methods to solve large scale systems is given by the *waveform relaxation method*. This method was first proposed by Lelarasmee, Ruehli and Sangiovanni–Vincentelli [27] for the time-domain analysis of large scale integrated circuits. For the waveform algorithm concerning deterministic processes and related aspects, many research papers can be found, e.g. Bremer and Schneider [4], Bremer [5], Burrage [6], in't Hout [12], Jackiewicz and Kwapisz [16], Jansen et al. [17], Jansen and Vandewalle [18], Leimkuhler [25, 26], Miekkala and Nevanlinna [30], Nevanlinna and Odeh [32], Sand and Burrage [36], Schneider [37, 38, 39], Ta'asan and Zhang [44], Zennaro [48], Zubik–Koval and Vandewalle [50], among many others.

In what follows we present a theoretical foundation for the construction and convergence of waveform iterations applied to systems of ordinary stochastic differential equations (SDEs) which are decomposable into weakly coupled subsystems. The attention is restricted to Itô-interpreted SDEs and $L^p$-solutions (i.e. strong solutions in the Banach space of $L^p(\Omega, \mathcal{F}, \mathbb{P})$-integrable random processes). For

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original works on stochastic integration, see Itô [13, 14, 15]. For basic aspects on the theory of SDEs in the spirit of Itô [13], see e.g. Arnold [1, 2], Dynkin [8], Gard [10], Khas'minskij [19], Krylov [22], Mao [28], Protter [34] and Revuz and Yor [35].

We see our main contribution in deriving precise bounds for the Lipschitz-constants of the related stochastic integral operator and in describing their influence on the $L^p$-convergence of waveform iteration methods depending on the splitting into subsystems. However, a qualitative comparison with other numerical techniques for stochastic differential equations (SDEs) is left to the interested reader.

The paper is organized as follows. In Section 2 we describe the key idea of waveform relaxation method. Section 3 presents a proof for the existence and uniqueness of an initial value problem for Itô-type stochastic differential equations (SDEs) using fixed point techniques on appropriate Banach spaces in order to derive conditions for the $L^p$-convergence of waveform relaxation methods with $p \geq 2$. Section 4 generalizes this idea to the case of one-sided Lipschitz-continuity of the drift part, restricted to drift coefficients satisfying an angle condition. An illustrative example is given in Section 5. Section 6 closes this paper with some concluding remarks.

## 2. The general idea of waveform relaxation methods

At first we convert the initial-value problem problem related to Itô-interpreted stochastic differential equations into a fixed point problem. Therefore, we can consider

$$x \;=\; \mathbb{T}x \,+\, g \tag{1}$$

where $\mathbb{T}$ maps the function space $\mathcal{U}$ into itself, and $g \in \mathcal{U}$. There are several techniques to find appropriate conditions on the operator $\mathbb{T}$ guaranteeing a unique solution $x^* \in \mathcal{U}$ of system (1) and resulting in an efficient algorithm to approximate $x^*$. In the case that (1) represents a network of weakly connected subsystems with quite different behavior, i.e. (1) carries the feature of a large scale system, the *waveform relaxation method* is an efficient approach to approximate $x^*$, formulated as follows:

(i) *Decomposition step*: Find a suitable representation of the space $\mathcal{U}$ as a product of subspaces $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n$, i.e.

$$\mathcal{U} \;=\; \mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_n\,, \tag{2}$$

and a corresponding splitting of $\mathbb{T}$ into $\mathbb{T}_1, \ldots, \mathbb{T}_n$ and $g$ into $g_1, ..., g_n$ such that the fixed point problem (1) is equivalent to the system

$$\begin{array}{rcl} x^{(1)} & = & \mathbb{T}_1(x^{(1)}, ..., x^{(n)}) + g_1, \\ x^{(2)} & = & \mathbb{T}_2(x^{(1)}, ..., x^{(n)}) + g_2, \\ \ldots\ldots & .. & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x^{(n)} & = & \mathbb{T}_n(x^{(1)}, ..., x^{(n)}) + g_n \end{array} \tag{3}$$

where $x^{(k)}, g_k \in \mathcal{U}_k$, and $\mathbb{T}_k$ maps $\mathcal{U}$ into the subspace $\mathcal{U}_k$ for $k = 1, 2, ..., n$.

(ii) *Solution step*: By an appropriate procedure, solve the $k$-th subsystem

$$x^{(k)} \;=\; \mathbb{T}_k(x^{(1)}, ..., x^{(k-1)}, x^{(k)}, x^{(k+1)}, ..., x^{(n)}) \;+\; g_k\,. \tag{4}$$

Here, $x^{(j)}, j = 1, 2, ..., n$ with $j \neq k$ are the inputs from other subsystems.

(iii) *Relaxation step*: Derive conditions such that the successive solution of subsystems (4) leads to the unique solution of the large scale system (of SDEs, specified later)