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ERROR ESTIMATES AND SUPERCONVERGENCE OF MIXED FINITE ELEMENT FOR QUADRATIC OPTIMAL CONTROL

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Abstract. In this paper we present a priori error analysis for mixed finite element approximation of quadratic optimal control problems. Optimal a priori error bounds are obtained. Furthermore super-convergence of the approximation is studied.

Key Words. optimal control, mixed finite element methods, error estimates, and superconvergence.

1. Introduction

Finite element approximation of optimal control problems plays a very important role among the numerical methods for these problems. The literature in this aspect is huge. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems; see, for example, [1], [2], [10], [14], [15], [24], and [32]. For optimal control problems governed by linear state equations, a priori error estimates of the standard finite element approximation were established long ago; see, for example, [8] and [23]. It is, however, much more difficult to obtain such error estimates for control problems where the state equations are nonlinear or where there are inequality state constraints. For a class of nonlinear optimal control problems with equality constraints, a priori error estimates were established in [12]. Some important flow controls are included in this class of problems. A priori error estimates have also been obtained for a class of state constrained control problems in [31], though the state equation is assumed to be linear. In [20] this assumption has been removed by reformulating the control problem as an abstract optimization problem in some Banach spaces and then applying nonsmooth analysis. In fact, the state equation there can be a variational inequality. Some recent progress in a priori error estimates can be found in [3], and in [18], [21] and [22] for a posteriori error estimates. Systematic introduction of the finite element method for PDEs and optimal control problems can be found in, for example, [5], [13], [26], and [30].

In many control problems, the objective functional contains gradient of the state variables. Thus accuracy of gradient is important in numerical approximation of the state equations. Traditionally in such cases mixed finite element methods should be used for discretisation of the state equations. In computational optimal control,

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mixed finite element methods are not as widely used as in engineering simulations. In particular there doesn't seem to exist much work on theoretical analysis of mixed finite element approximation of optimal control problems in the literature.

In this paper we study error estimates and super-convergence of mixed finite element schemes for quadratic optimal control problems. The problem that we are interested in is the following optimal control problem:

$$\min_{u\in K\subset L^2(\Omega_U)} \frac{1}{2}\left\{\int_\Omega |\boldsymbol{p}-\boldsymbol{p}_0|^2 + \int_\Omega (y-y_0)^2 + \int_{\Omega_U} u^2\right\} \tag{1.1}$$

$$\mathrm{div}\boldsymbol{p} = f + Bu \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{p} = -A\nabla y, \qquad \text{in } \Omega, \tag{1.3}$$

$$y = 0, \qquad \text{on } \partial\Omega, \tag{1.4}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$ is a convex polygon or has smooth boundary $\partial\Omega$, Ω_U is a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega_U$, K is a closed convex set in $L^2(\Omega_U)$. Further specifications on data will be given later. The coefficient matrix $A \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$ is symmetric and uniformly elliptic, i.e., $A(x)$ is a symmetric and positive definite 2×2-matrix, with eigenvalues $\lambda_j(x) \in \mathbb{R}$ satisfying

$$0 < c_A \le \lambda_1(x),\ \lambda_2(x) \le C_A \tag{1.5}$$

for almost all $x \in \Omega$.

In this paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $||\cdot||_{m,p}$ given by $||\phi||^p_{m,p} = \sum_{|\alpha|\le m} ||D^\alpha\phi||^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $||\phi||^p_{m,p} = \sum_{|\alpha|=m} ||D^\alpha\phi||^p_{L^p(\Omega)}$. We set $W^{m,p}_0(\Omega) = \{\phi \in W^{m,p}(\Omega) : \phi|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ and $||\cdot||_m = ||\cdot||_{m,2}$. In addition C denotes a general positive constant independent of h.

2. Mixed finite element approximation of optimal control problems

Let

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2,\ \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}, \tag{2.1}$$

endowed with the norm given by

$$||\boldsymbol{v}||_{\mathrm{div}} = ||\boldsymbol{v}||_{H(\mathrm{div};\Omega)} = \left(||\boldsymbol{v}||^2_{0,\Omega} + ||\mathrm{div}\boldsymbol{v}||^2_{0,\Omega}\right)^{1/2},$$

and

$$W = L^2(\Omega). \tag{2.2}$$

We denote

$$U = L^2(\Omega_U). \tag{2.3}$$

To consider the mixed finite element approximation of our optimal control problems, we need a weak formulation for the state equation (1.2)-(1.4). We recast (1.1)-(1.4) in the following weak form: (CCP) find $(\boldsymbol{p}, y, u) \in \boldsymbol{V}\times W\times U$ such that

$$\min_{u\in K\subset L^2(\Omega_U)} \frac{1}{2}\left\{\int_\Omega |\boldsymbol{p}-\boldsymbol{p}_0|^2 + \int_\Omega (y-y_0)^2 + \int_{\Omega_U} u^2\right\} \tag{2.4}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.5}$$

$$(\mathrm{div}\boldsymbol{p}, w) = (f + Bu, w), \qquad \forall\ w \in W, \tag{2.6}$$