INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 3, Pages 322–332

SUPERCONVERGENCE STUDIES OF QUADRILATERAL NONCONFORMING ROTATED Q_1 ELEMENTS

PINGBING MING, ZHONG-CI SHI AND YUN XU

(Communicated by Zhimin Zhang)

Abstract. For the nonconforming rotated Q_1 element over a mildly distorted quadrilateral mesh, we propose a superconvergence property at the element center, the vertices and the midpoints of four edges. Numerics are presented to confirm this observation.

Key Words. Superconvergence, Nonconforming rotated Q_1 element, Kershaw mesh.

1. Introduction

Nonconforming rotated Q_1 element [21] (NRQ_1) with mean integral over edges as degrees of freedom (NRQ_1^a) has been widely used in several fields including the computational fluids [21, 25], the crystalline microstructure [11, 13], the Chapman-Ferraro problem [12], the Reissner-Mindlin plate bending problem [16], and the streamline-diffusion problem [22, 24]. Compared with the standard bilinear element, NRQ_1^a exhibits better stability in these problems.

A new NRQ_1 element introduced by Ming and Shi leads to a truly locking-free Reissner-Mindlin plate element over general quadrilateral meshes [19]. Compared to NRQ_1^a, this element has an extra degree of freedom (we call it the five-point NRQ_1, see Definition 2.3). A similar element was presented in [4] to approximate Navier-Stokes equations.

The convergence rate in the energy norm of both NRQ_1 elements is of first order over a rectangular mesh [11, 21]. As to the general quadrilateral mesh, the five-point NRQ_1 retains the first order convergence rate, while NRQ_1^a converges with first order if the mesh is mildly distorted [15, 17]. An example is given to show the first order optimality [15].

Meanwhile, a superconvergence property at element center on the rectangular parallelopiped mesh was obtained for NRQ_1^a [11]. For the mildly distorted quadrilateral mesh, we proved [20] that the supercovergence property is valid not only for the element center, but also for the vertices and midpoints of four edges. Therefore, both elements share the same superconvergence points as the bilinear element [5]. Extensive numerics will be presented in this paper to confirm the theoretic prediction. The same phenomenon was also numerically observed for another NRQ_1 element that employs midpoints value of each edge as degrees of freedom, however,

Received by the editors July 31, 2004 and, in revised form, March 30, 2005.

2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

This research Ming was supported by Research Foundation Of Education Department 6758700 and partly supported by Chinese Natural Science Foundation through 10201033.

there is no theoretic support up to now. Some superclose results for the rectangular NRQ_1 element and its variants can be found in [2, 14, 15, 22].

The outline of this paper is as follows. In the next section, we introduce NRQ_1^a, NRQ_1^p, and the five-point NRQ_1 element, the quadrilateral mesh conditions. The main results are stated in § 3. Numerical results and discussion are given in the last section.

2. Nonconforming Rotated Q_1 Element

For any convex polygon Ω, we use the standard Sobolev space $W^{k,p}(\Omega)$ [1]. Denote by $⨍_{\Omega_1} f$ the mean value of a function f over the sub-domain Ω_1 of Ω.

We consider the general second order elliptic boundary value problem

$$\begin{cases} -\partial_x\big(a_{11}\partial_x u\big) - \partial_x\big(a_{12}\partial_y u\big) - \partial_y\big(a_{21}\partial_x u\big) - \partial_y\big(a_{22}\partial_y u\big) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\{a_{ij}\}_{i,j=1}^2 \in W^{2,\infty}(\Omega)$, and

$$\lambda|\,\xi\,|^2 \le \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \le \Lambda|\,\xi\,|^2 \quad \text{for all} \ \ \xi \in \mathbb{R}^2.$$

Let $\mathcal{T}_h$ be a partition of $\overline{\Omega}$ by convex quadrilaterals K with the mesh size h_K and $h := \max_{K\in\mathcal{T}_h} h_K$. We assume that $\mathcal{T}_h$ is shape regular in the sense of Ciarlet-Raviart [6, p. 247]. Namely, all quadrilaterals are convex and there exist constants $\rho_1 \ge 1$ and $0 < \rho_2 < 1$ such that

$$h_K/\underline{h}_K \le \rho_1, \qquad |\cos\theta_{i,K}\,| \le \rho_2, \quad i = 1,2,3,4 \quad \text{for all} \ \ K \in \mathcal{T}_h.$$

Here h_K, $\underline{h}_K$ and $\theta_{i,K}$ denote the diameter, the shortest length of sides, and the interior angles of K, respectively.

We introduce a mesh condition which quantifies the deviation of a quadrilateral from a parallelogram.

Definition 2.1. $(1+\alpha)$-section condition $(0 \le \alpha \le 1)$ [18] *The distance d_K between the midpoints of two diagonals of $K \in \mathcal{T}_h$ is of order $\mathcal{O}(h_K^{1+\alpha})$ uniformly for all elements K as $h \to 0$.*

The extreme case $\alpha = 0$ represents an unstructured quadrilateral mesh subdivision. The mesh partition in Fig. 4 is a particular one, which consists of trapezoids generating from a typical trapezoid with translation and dilation. In case of $\alpha = 1$, the mesh satisfies the *Bi-section condition* [23], which is also the 1-strongly regular mesh [27].

Definition 2.2. *For every element $K \in \mathcal{T}_h$, we call K satisfies the $(1+\beta_K)$-uniform condition if for every elements $K^* \in S(K)$, there exist constants $\beta_1(K^*)$ and $\beta_2(K^*)$ such that*

$$\begin{aligned} |\,\overrightarrow{M_1M_3} - \overrightarrow{M_3M_6}\,| &= \mathcal{O}(h_K^{1+\beta_1(K^*)} + h_{K^*}^{1+\beta_1(K^*)}), \\ |\,\overrightarrow{M_2M_4} - \overrightarrow{M_5M_7}\,| &= \mathcal{O}(h_K^{1+\beta_2(K^*)} + h_{K^*}^{1+\beta_2(K^*)}). \end{aligned} \tag{2.2}$$

We define β_K as

$$\beta_K\!: = \min_{K^*\in S(K)} \min(\beta_1(K^*), \beta_2(K^*)),$$

where $S(K)$ is the subset of $\mathcal{T}_h$ with nonempty intersection with $\overline{K}$, and we refer to FIG. 1 *for M_1M_3, M_3M_6 and M_2M_4, M_5M_7.*