INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 3, Pages 333–347

RELIABLE AND EFFICIENT AVERAGING TECHNIQUES AS UNIVERSAL TOOL FOR A POSTERIORI FINITE ELEMENT ERROR CONTROL ON UNSTRUCTURED GRIDS

CARSTEN CARSTENSEN

(Communicated by Zhimin Zhang)

Abstract. The striking simplicity of averaging techniques in a posteriori error control of finite element methods as well as their amazing accuracy in many numerical examples over the last decade have made them an extremely popular tool in scientific computing. Given a discrete stress or flux p_h and a post-processed approximation $A(p_h)$, the a posteriori error estimator reads $\eta_A := \|p_h - A(p_h)\|$. There is not even a need for an equation to compute the estimator η_A and hence averaging techniques are employed everywhere. The most prominent example is occasionally named after Zienkiewicz and Zhu, and also called gradient recovery but preferably called averaging technique in the literature.

The first mathematical justification of the error estimator η_A as a computable approximation of the (unknown) error $\|p - p_h\|$ involved the concept of superconvergence points. For highly structured meshes and a very smooth exact solution p, the error $\|p - A(p_h)\|$ of the post-processed approximation Ap_h may be (much) smaller than $\|p - p_h\|$ of the given p_h. Under the assumption that $\|p - A(p_h)\| =$ h.o.t. is in relative terms sufficiently small, the triangle inequality immediately verifies reliability, i.e.,

$$\|p - p_h\| \le C_{\text{rel}}\, \eta_A + \text{ h.o.t.},$$

and efficiency, i.e.,

$$\eta_A \le C_{\text{eff}}\, \|p - p_h\| + \text{ h.o.t.},$$

of the averaging error estimator η_A. However, the required assumptions on the symmetry of the mesh and the smoothness of the solution essentially contradict the use of adaptive grid refining when p is singular and the proper treatment of boundary conditions remains unclear.

This paper aims at an actual overview on the reliability and efficiency of averaging a posteriori error control for unstructured grids. New aspects are new proofs of the efficiency of *all* averaging techniques and for *all* problems.

Key Words. a posteriori error estimate, efficiency, finite element method, gradient recovery, averaging operator, mixed finite element method, non-conforming finite element method

1. Overview

The outcome of a first-order finite element method (FEM) is a globally continuous and piecewise polynomial function u_h. The corresponding flux or stress p_h is

Received by the editors November 1, 2004 and, in revised form, May 1, 2005.
2000 *Mathematics Subject Classification.* 65N30.

usually a linear operator $\mathbb{C}$ evaluated for the gradient Du_h (or its symmetric part) of the finite element function u_h,

$$p_h := \mathbb{C}Du_h \in P_0(\mathcal{T};\mathbb{M}).$$

Here and throughout, $\mathcal{T}$ is a triangulation of the computational domain Ω, $\mathbb{M}$ is a space of vectors or matrices, and $P_k(\mathcal{T};\mathbb{M})$ denotes the piecewise polynomials of degree $\leq k$ [piecewise with respect to $\mathcal{T}$ and with values in $\mathbb{M}$].

Typical examples are elliptic partial differential equations of second order in Ω, namely,

$$-\operatorname{div} \mathbb{C}\, Du = f \quad \text{in } \Omega,$$

for the Poisson or Lamé equations, which give rise to a weak formulation

$$a(u,v) = b(v) \quad \text{for all } v \in V.$$

Here and throughout, a is a bounded bilinear form on the Hilbert space V (or on some larger space) and b is a bounded linear functional on V, written $b \in V^*$.

For the ease of this overview, the presentation is restricted to homogeneous Dirichlet conditions on the entire boundary. Then, the flux or stress $p := \mathbb{C}Du$ satisfies no prescribed boundary conditions and can be approximated by some globally continuous and piecewise polynomial functions which form a discrete space

$$\mathcal{Q}_h := P_1(\mathcal{T};\mathbb{M}) \cap C^0(\Omega;\mathbb{M}).$$

Given p_h, the norm $\|\cdot\|$, and the discrete space $\mathcal{Q}_h$, the minimal averaging a posteriori error estimator η_M reads

$$\eta_M := \min_{q_h \in \mathcal{Q}_h} \|p_h - q_h\|.$$

The computation of η_M involves a global minimization which can be solved by an iterative scheme which is not too costly in many applications when a (weighted) L^2 projection is involved. However, local versions appear as accurate as η_M which involves a postprocessing defined by an operator

$$A : Q \to \mathcal{Q}_h \quad \text{for } Q := \{\mathbb{C}Dv : v \in V\} \subset L := L^2(\Omega;\mathbb{M}).$$

Then the A *averaging a posteriori error estimator* η_A reads

$$\eta_A := \|p_h - A(p_h)\|.$$

One particular important example is the ZZ estimator [24]

$$\eta_Z := \|p_h - Z(p_h)\| \approx \eta_{\mathcal{E}},$$

which is equivalent to the jumps across interior element edges $\eta_{\mathcal{E}}$ (for conforming P_1 FEM). Details on the notation follow in Section 2. It is obvious that $\eta_M \leq \eta_A$. The surprising converse of which will be shown below for a class of averaging operators. In fact, Section 2 establishes

$$\eta_A \approx \eta_M \approx \eta_Z \approx \eta_{\mathcal{E}}.$$

Here and throughout, the statement $a \lesssim b$ abbreviates $a \leq C\, b$ for some positive generic constant C which does not depend on the meshsize in $\mathcal{T}$, and $a \lesssim b \lesssim a$ is abbreviated by $a \approx b$.

This paper discusses examples of problems and estimators and studies their *reliability*, i.e.,

$$\|p - p_h\| \lesssim \eta_M + \text{ h.o.t.},$$

(recall that h.o.t. abbreviates higher-oder terms, the meaning of which is clarified below) and their *efficiency*, i.e.,

$$\eta_M \lesssim \|p - p_h\| + \text{ h.o.t.}$$